# Extension of a theorem of Wschebor to free and matrix Brownian motions

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**Abstract**

In 1992, M. Wschebor proved a theorem on the convergence of small increments of the Brownian motion. Since then, it has been extended to various processes. We prove a version of this theorem for the Hermitian Brownian motion and the free Brownian motion. Since these theorems deal with a convergence to a deterministic limit, we prove also the convergence in distribution of the corresponding fluctuations.

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## 1 Introduction

In 1992 [23], Mario Wschebor proved the following remarkable property of the linear Brownian motion $(W(t), t \geq 0\ ; W(0) = 0)$. If

$$\mathcal{W}^\varepsilon := \varepsilon^{-1/2}\left(W(\cdot + \varepsilon) - W(\cdot)\right) \tag{1.1}$$

and if $\lambda$ is the Lebesgue measure on $[0,1]$, then, almost surely, for every $x \in \mathbb{R}$ and every $t \in [0,1]$:

$$\lim_{\varepsilon \to 0} \lambda\{s \leq t : \mathcal{W}^\varepsilon(s) \leq x\} = t\Phi(x)\,, \tag{1.2}$$

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where $\Phi$ is the distribution function of the standard normal distribution $\mathcal{N}(0;1)$. Motivated by the study of crossings of stochastic processes in continuous time, he extended this result to more general smooth approximations of trajectories.

Let $(W_t, t \in \mathbb{R})$ be the bilateral Brownian motion, i.e. $(W_t, t \geq 0)$ and $(W_{-t}, t \leq 0)$ are inedependent Brownian motions starting from 0.

**Theorem 1.1** ([23], [4], [25]). *Let $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support and bounded variation satisfying $\|\varphi\|_2 = 1$ and*

$$W_\varphi^\varepsilon(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\varepsilon}} \varphi\left(\frac{t-s}{\varepsilon}\right) dW(s)\,, \tag{1.3}$$

*and let*

$$\mu^\varepsilon = \int_0^1 \delta_{W_\varphi^\varepsilon(t)} dt\,, \tag{1.4}$$

*be the corresponding occupation measure. Then almost surely as $\varepsilon \to 0$, $\mu_\varepsilon$ converges to the standard normal distribution $\mathcal{N}(0,1)$ for the convergence of moments, i.e. that for every $k$*

$$\int x^k d\mu^\varepsilon(x) = \int_0^1 (W_\varphi^\varepsilon(t))^k dt \underset{\varepsilon \to 0}{\longrightarrow} \mathbb{E}(\mathcal{N}^k)\,, \tag{1.5}$$

*where $\mathcal{N} \overset{(d)}{=} \mathcal{N}(0;1)$.*

It is a statement which we call law of large numbers (LLN). Actually (1.1) corresponds to $\varphi = 1_{[-1,0]}$.

Since the Brownian motion $W$ is self-similar (Property P1) and has stationary increments (P2), it is possible to reduce the study of $\mu^\varepsilon$ $(\varepsilon \to 0)$ to the study of the occupation measure in large time $(T := \varepsilon^{-1} \to \infty)$ for the rescaled process

$$W_\varphi^1(t) = \int_{-\infty}^{\infty} \varphi(t-s) dW(s)$$

which does not depend on $\varepsilon$. This moving average process is stationary Gaussian (it is the Slepian process when $\varphi = 1_{[-1,0]}$). Moreover, the independence of increments of $W$ (P3) induces a finite-dependence for $W_\varphi^1$ (recall that $\varphi$ is compactly supported), so $W_\varphi^1$ is ergodic which allows to invoke Birkhoff's theorem.

Later the above result has been extended to other types of processes (sharing properties P1 and P2), in particular the fractional Brownian motion

([24], [4], [25]). The Gaussian character of this process allows the use of its spectral measure instead of the independence of increments.

Corresponding fluctuations have been established :

$$\frac{1}{\sqrt{\varepsilon}} \int (g - \mathbb{E}g(\mathcal{N}))\, d\mu^\varepsilon \xrightarrow[\varepsilon \to 0]{(d)} \sigma(g)\mathcal{N} \tag{1.6}$$

for a large class of functions $g$ and where $\sigma(g)$ is an explicit function of $g$ (see [7] and [25] for a very complete review). The main tool is the Hermite polynomials.

In the present paper, we extend these results to the matricial Brownian motion and to the free Brownian motion. For the LLN, the solution is straightforward. For matrix fluctuations, we need a convenient notion of Hermite polynomial and for the free fluctuations we will use Tchebyshev polynomials. In the scalar and free cases, these fluctations are a consequence of variations of the Breuer-Major theorem.

In Sec. 2 we recall the scalar results, presenting short proofs to prepare the way for proofs in the free and matricial extensions. In Sec. 3 we establish the LLN and fluctuations for the free case, since the treatment is rather easy. The more involved matrix case is handled in Sec. 4 and 5. Eventually, we consider in Sec. 6 a different model of fluctuations using the Hermite matrix-variate polynomials.

# 2 The scalar case

## 2.1 LLN . Proof of Theorem 1.1.

There are two proofs of the LLN, the historical one and the "ergodic" one. For the sake of completeness we recall the proof of (1.5) given by Wschebor.

Let us asssume that the support of $\varphi$ is contained in $[-a, a]$. The first step consists in stating that all moments of $\mu_{\mathcal{W}^\varepsilon}$ converge in $L^2$ to the corresponding moments of $\mathcal{N}$. The marginals of $W^\varepsilon_\varphi$ are standard normal, hence, for every $k$

$$\mathbb{E}\left(\int_0^1 W^\varepsilon_\varphi(t)^k dt\right) = \int_0^1 \mathbb{E}\left(W^\varepsilon_\varphi(t)^k\right) dt = \mathbb{E}\left(\mathcal{N}^k\right). \tag{2.1}$$

Besides

$$\begin{aligned}\mathrm{Var}\left(\int_0^1 W^\varepsilon_\varphi(t)^k dt\right) &= \int_{[0,1]^2} \mathrm{Cov}\left(W^\varepsilon_\varphi(t)^k, W^\varepsilon_\varphi(s)^k\right) ds dt \\ &= \varepsilon^2 \int_{[0,1/\varepsilon]^2} \mathrm{Cov}\left(W^1_\varphi(t)^k, W^1_\varphi(s)^k\right) ds dt\,,\end{aligned}$$

where in the second identity we use the scaling property

$$(W^\varepsilon_\varphi(\varepsilon t), 0 \le t \le 1) \overset{(d)}{=} (W^1_\varphi(t), 0 \le t \le 1/\varepsilon)\,. \tag{2.2}$$

Then, we split the domain of integration into $\{(s,t) \in [0, 1/\varepsilon]^2 : |t-s| > 2a\}$ and $[0, 1/\varepsilon]^2 \cap |t-s| \le 2a$. The first integral is zero since the process $W^1_\varphi$ is $a$-dependent, and the second integral is bounded, owing to Cauchy-Schwarz by

$$\lambda\{(s,t) \in [0, 1/\varepsilon]^2 : |t-s| \le 2a\}\mathrm{Var}(\mathcal{N}^k) = O(1/\varepsilon)\,.$$

Then Borel-Cantelli and the properties of the Brownian paths allow to get an a.s. convergence. We will not give details here.

## 2.2 Fluctuations

The fluctuations in the classical scalar case are a consequence of the continuous version of the Breuer-Major theorem . Let us first recall that if $F$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathbb{E}F(\mathcal{N})^2 < \infty$, it has the Hermite expansion

$$F(x) = \sum_{n=0}^\infty c_n H_n(x)$$

where $H_n$ is the Hermite polynomial of degree $n$. The Hermite rank of $F$ is the smallest $n$ such that the Hermite coefficient $c_n$ is non-zero.

**Theorem 2.1.** *[5] [9, Th. 1.1] Let $X$ be a Gaussian stationary process with covariance function $\rho$. Let us assume that $F$ is such that $\mathbb{E}F(\mathcal{N})^p < \infty$ for some $p \ge 2$ and that its Hermite rank is $\ell \ge 1$. Suppose also that $\int |\rho(t)|^\ell dt < \infty$. Then the family of processes*

$$Z_\varepsilon(t) := \varepsilon^{1/2} \int_0^{t/\varepsilon} F(X_s)ds\ , t \ge 0 \tag{2.3}$$

*converges, as $\varepsilon$ tends to zero to $(\sigma_W(F) W_t, t \ge 0)$, where*

- *$(W_t, t \ge 0)$ is a standard Brownian motion,*
- $\sigma^2_W(F) = 2\sum_{q=\ell}^\infty c_q^2 q! \int_0^\infty \rho(t)^q dt$
- *the convergence in law is fidi if $p = 2$ and in $\mathcal{C}(\mathbb{R}^+)$ if $p > 2$.*

In the Wschebor original case,

$$\varphi = 1_{]-1,0]}\ , \rho(t) = (1 - |t|)_+\ , \int_0^\infty \rho(t)^q dt = 1/(q+1)\,.$$

**Corollary 2.2.** *If $F$ is such that $\mathbb{E}F(\mathcal{N}^p) < \infty$ for some $p \geq 2$ and its Hermite rank is $\ell \geq 1$, then*

$$\left(\varepsilon^{-1/2} \int_0^t F(W_\varphi^\varepsilon(s)ds \ , t \geq 0\right) \xrightarrow[\varepsilon \to 0]{(d)} \left(\sigma_W^2(F) W_t \ , t \geq 0\right) \tag{2.4}$$

*in the above conditions with*

$$\rho(u) = \int_{-\infty}^{\infty} \varphi(u+\tau)\varphi(\tau)d\tau \,. \tag{2.5}$$

Notice that $\rho$ is defined on $\mathbb{R}$, even and satisfy

$$\rho(u) \leq 1 \,, \tag{2.6}$$

(use $\|\varphi\|_2 = 1$ and Cauchy-Schwarz). All along the sequel we will use the quantity

$$\sigma_k^2 = \int_{-\infty}^{\infty} \rho(t)^k dt \,, \tag{2.7}$$

so that in the above (2.4)

$$\sigma_W^2(F) = \sum_{\ell}^{\infty} q! c_q^2 \sigma_q^2 \,. \tag{2.8}$$

In [6] there are two proofs of the convergence of one-dimensional marginals in Th. 2.1 : a classical one and a modern one based on the Fourth Moment Theorem. Both are based on a spectral representation of the process.

We give here a variation of the proof of Corollary 2.2, starting from the moving average representation itself and multiple Wiener-Ito integrals.

We will begin with a fixed chaos, i.e. $F = H_n$, a fixed time $t = 1$ in (2.3) and $X_s = W_\varphi^1(s)$. We set $\varepsilon = T^{-1}$ . We have

$$T^{-1/2} \int_0^T H_n\left(W_\varphi^1(t)\right) dt = I_n^W(f_T) \tag{2.9}$$

where

$$f_T(t_1, \ldots, t_n) = T^{-1/2} \int_0^T \varphi(t - t_1) \cdots \varphi(t - t_n) dt \,, \tag{2.10}$$

and for $f$ symmetric function in $L^2(\mathbb{R}_+^n)$

$$I_n^W(f) = \int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n) dW_{s_1} \cdots dW_{s_n} \,. \tag{2.11}$$

This yields

$$\mathbb{E}\left(I_n^W(f_T)\right)^2 = n!\|f_T\|^2 = n!T^{-1}\int\left(\int_0^T \varphi(t-t_1)\cdots\varphi(t-t_n)dt\right)^2 dt_1\cdots dt_n$$
$$= n!T^{-1}\int_{[0,T]^2}\left(\int \varphi(t-t_1)\varphi(s-t_1)dt_1\right)^n dtds$$
$$= n!T^{-1}\int_{[0,T]^2}\rho(t-s)^n dtds\,,$$

where we used Fubini. After a change of variable, we conclude that

$$\mathbb{E}\left(I_n^W(f_T)\right)^2 = 2n!\int_0^T\left(1-\frac{u}{T}\right)\rho(u)^n du$$
$$\underset{T\to\infty}{\longrightarrow} 2n!\int_0^\infty \rho(u)^n du = n!\sigma_n^2\,. \qquad (2.12)$$

We apply the Fourth Moment Theorem ([21, Th. 1]) which says that the convergence in distribution of $I_n^W(f_T)$ to a normal variable as $T\to\infty$ is equivalent to the convergence of the fourth moments and also equivalent to the convergence in $L^2$ to zero of contractions

$$f_T\otimes_k f_T(\xi_1,\ldots,\xi_{2n-2k}) =$$
$$= \int f_T(s_1,\ldots,s_k,\xi_1,\ldots,\xi_{n-k})f_T(s_1,\ldots s_k,\xi_{n-k+1},\ldots,\xi_{2n-2k})ds_1\ldots ds_k\,,$$

for $k=1,\ldots,n-1$. But

$$\|f_T\otimes_k f_T\|^2 = T^{-2}\int_{[0,T]^4}\rho(t-\tau)^k\rho(s-\sigma)^k\rho(t-s)^{n-k}\rho(\tau-\sigma)^{n-k}dtdsd\tau d\sigma\,,$$

which tends to 0 (see p.11-12 in [6]). This proves

$$I_n^W(f_T)\xrightarrow[T\to\infty]{(d)}\mathcal{N}(0,\sigma_n^2)\,. \qquad (2.13)$$

In view of proving the fidi convergence of the process

$$T^{-1/2}\int_0^{Tu} H_n\left(W_\varphi^1(t)\right)dt\ ,\ u\geq 0$$

to the Brownian motion, we apply [22, Th. 1]. We have just to check that if for $a<b$

$$f_T^{[a,b]}(s_1,\ldots,s_n) := \frac{1}{\sqrt{T}}\int_{aT}^{bT}\varphi(t-s_1)\cdots\varphi(t-s_n)dt$$

then for $u_1 < u_2 \leq u_3 < u_4$

$$\mathbb{E}\left(I_n^W(f_T^{[u_1,u_2]})I_n^W(f_T^{[u_3,u_4]})\right) \underset{T\to\infty}{\longrightarrow} 0 \tag{2.14}$$

but this expectation is equal to

$$\begin{aligned} n!\langle f_T^{[u_1,u_2]}, f_T^{[u_3,u_4]}\rangle_{L^2} &= \frac{n!}{T}\int_{[Tu_1,Tu_2]\times[Tu_3,Tu_4]} \rho(t-\tau)^n dt d\tau \\ &= \frac{n!}{T}\int_{\tau\in[Tu_1,Tu_2], v+\tau\in[Tu_3,Tu_4]} \rho(v)^n dv d\tau \\ = n! \int \lambda\left([u_1,u_2]\cap[u_3-vT^{-1}, u_4-vT^{-1}]\right)\rho(v)^n dv\,, & \end{aligned} \tag{2.15}$$

where $\lambda$ is the Lebesgue measure. It is straightforward to see that the limit is 0. This proves the fidi convergence of increments.

It is straightforward to extend this result to obtain the convergence for a combination of Hermite polynomials. Eventually tightness is proved in [9].

*Remark* 1. The above results may be extended to moving average processes driven by more general processes sharing with Brownian motion properties P1 and P2 (see Sec. 1), like Hermite processes (see [16]).

## 3 The free case

A non-commutative probability space is an algebra $\mathcal{A}$ of operators on a complex separable Hilbert space, closed under adjoint and convergence in the weak operator topology equipped with a trace $\tau$, that is, a linear functional weakly continuous, satisfying:

- $\tau(\mathbf{1}) = 1$
- $\tau(ab) = \tau(ba)$
- $\tau(aa^*) \geq 0$ and $\tau(aa^*) = 0$ iff $a = 0$.

The self-adjoint elements of $\mathcal{A}$ are called non-commutative random variables. If $a$ is such a random variable, the linear form on the set of polynomials defined by

$$P \in \mathbb{C}[X] \mapsto \tau(P(a))\,,$$

is called the distribution of $a$. In this case there exists a unique probability measure $\mu$ such that

$$\tau(P(a)) = \int_{\mathbb{R}} P(x)d\mu(x)\,.$$

The semi-circle distribution of variance $\sigma^2$ is

$$\mathrm{SC}(0;\sigma^2)(dx) = \frac{1}{2\pi\sigma^2}\sqrt{(4\sigma^2 - x^2)_+}\ dx\,.$$

If $a$ is a non-commutative SC(0; 1) (SC for short) random variable, its moments are given by

$$\tau(a^{2p}) = \frac{1}{p+1}\binom{2p}{p} = C_p\ , \tau(a^{2p+1}) = 0\,, \tag{3.1}$$

where $C_p$ is the $p$th Catalan number. In particular $\tau(a^2) = 1$ and $\tau(a^4) = 2$.

The free Brownian motion is a family of $(\mathbf{S}(t), t \geq 0)$ of elements of $\mathcal{A}$ such that:

- $\tau(\mathbf{S}(t)) = 0$ for all $t$
- $\mathbf{S}$ has free increments
- $\mathbf{S}(0) = 0$ and the distribution of $\mathbf{S}(t) - \mathbf{S}(s)$ is $\mathrm{SC}(0; t - s)$ for $s < t$.

As in Sec. 1, we can define also a bilateral free Brownian motion, i.e. $(\mathbf{S}(t), t \geq 0)$ and $(\mathbf{S}(-t), t \leq 0)$ are free Rownina motion which are mutually free.

Such a process is self-similar with index 1/2. Now we define

$$S^\varepsilon_\varphi(t) = \varepsilon^{-1/2}\int_{-\infty}^{\infty} \varphi\left(\frac{t-s}{\varepsilon}\right) d\mathbf{S}(s)\,, \tag{3.2}$$

where as usual, we assume that the kernel $\varphi$ is bounded, has a support included in $[-a, a]$ and satisfies $\|\varphi\|_2 = 1$. In all the sequel, $P$ will be a polynomial in one variable.

## 3.1 LLN

**Proposition 3.1.** *For the convergence of moments, we have:*

$$\lim_\varepsilon \int_0^1 P(\mathbf{S}^\varepsilon_\varphi(t))dt = \int_0^1 P(x)d\,\mathrm{SC}(x)\cdot \mathbf{1}\,. \tag{3.3}$$

*Proof.* It is clear that we may suppose that $P(x) = x^k$. We have to prove that for every $j$

$$\lim_{\varepsilon\to 0} \tau\left(\int_0^1 \left(\mathbf{S}^\varepsilon_\varphi(t)\right)^k dt\right)^j = (m_k)^j\,, \tag{3.4}$$

where $m_k = \tau(a^k)$ when $a$ is a SC random variable.

We adapt the proof of the scalar case Sec. 2.1. The basic properties are scaling and stationarity :

$$(\mathbf{S}^{\varepsilon}_{\varphi}(\varepsilon t), 0 \le t \le 1) \overset{(d)}{=} (\mathbf{S}^{1}_{\varphi}(t), 0 \le t \le 1/\varepsilon) \tag{3.5}$$

and stationarity, in particular

$$\mathbf{S}^{\varepsilon}_{\varphi}(t) \overset{(d)}{=} \mathrm{SC}\,. \tag{3.6}$$

Let us first prove (3.4) for $j = 1$ and $j = 2$.

For $j = 1$ there is no limit to take, since from (3.6)

$$\tau\left(\int_0^1 \left(\mathbf{S}^{\varepsilon}_{\varphi}(t)\right)^k\right) dt = \int_0^1 \tau\left(\left(\mathbf{S}^{\varepsilon}_{\varphi}(s)\right)^k\right) ds = m_k\,.$$

For $j = 2$, it is enough to prove

$$\lim_{\varepsilon} \tau\left(\int_0^1 \left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k\right)\right) dt\right)^2 = 0\,. \tag{3.7}$$

We have

$$\tau\left(\int_0^1 \left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k\right)\right) dt\right)^2 =$$

$$\begin{aligned}
&= \tau\left(\int_{[0,1]^2} \left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k\right)\right)\left(\mathbf{S}^{\varepsilon}_{\varphi}(s)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(s)^k\right)\right) dtds\right) \\
&= \int_{[0,1]^2} \tau\left(\left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(t)^k\right)\right)\left(\mathbf{S}^{\varepsilon}_{\varphi}(s)^k - \tau\left(\mathbf{S}^{\varepsilon}_{\varphi}(s)^k\right)\right)\right) dtds \\
&= \varepsilon^2 \int_{[0,1/\varepsilon]^2} \tau\left(\left(\mathbf{S}^{1}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(t)^k\right)\right)\left(\mathbf{S}^{1}_{\varphi}(s)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(s)^k\right)\right)\right) dtds\,.
\end{aligned}$$

As in the scalar case we split the domain of integration into $\{(s,t) \in [0,1/\varepsilon]^2 : |t-s| > 4a\}$ and $[0,1/\varepsilon]^2 \cap |t-s| \le 4a$. The first integral is zero since when $|t-s| > 4a$, $\mathbf{S}^{1}_{\varphi}(t)^k$ and $\mathbf{S}^{1}_{\varphi}(s)^k$ are free.

In the second integral we have by Cauchy-Schwarz

$$|\tau\left(\left(\mathbf{S}^{1}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(t)^k\right)\right)\left(\mathbf{S}^{1}_{\varphi}(s)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(s)^k\right)\right)\right)| \le$$

$$\le \sqrt{\tau\left(\mathbf{S}^{1}_{\varphi}(t)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(t)^k\right)^2\right)}\sqrt{\tau\left(\mathbf{S}^{1}_{\varphi}(s)^k - \tau\left(\mathbf{S}^{1}_{\varphi}(s)^k\right)^2\right)} = m_{2k} - (m_k)^2\,,$$

so that the second integral is

$$O\left(\lambda\{(s,t)\in[0,1/\varepsilon]^2 : |t-s|\le 4a\}\right) = O(1/\varepsilon)\,,$$

and then, after multiplication by $\varepsilon^2$ its contribution vanishes.

Assume $j\ge 3$. Setting

$$\mathcal{J}_j^\varepsilon = \left(\int_0^1 \left(\mathbf{S}_\varphi^\varepsilon(s)\right)^k ds\right)^j\,,$$

we have

$$\mathcal{J}_j^\varepsilon - (m_k)^j = (\mathcal{J}_1^\varepsilon - m_k)\Delta_j^\varepsilon\,, \tag{3.8}$$

where $\Delta_j^\varepsilon = \Sigma_{r=0}^{j-1}\mathcal{J}_r^\varepsilon (m_k)^{j-r}$ and by Cauchy-Schwarz again

$$\left|\tau(\mathcal{J}_j^\varepsilon) - (m_k)^j\right| \le \sqrt{\tau(\mathcal{J}_1^\varepsilon - m_k)^2}\sqrt{\tau((\Delta_j^\varepsilon)^2)} \tag{3.9}$$

Now since the stationary law SC has a compact support $[-2,2]$, we have for every $p$

$$\|\mathcal{J}_p^\varepsilon\| \le 2^{kp}\,, \tag{3.10}$$

so that, for every $\varepsilon$

$$\tau((\Delta_j^\varepsilon)^2) \le j^2 2^{2kj} \tag{3.11}$$

and then

$$\tau(\mathcal{J}_j^\varepsilon) - (m_k)^j = O\left(\sqrt{\tau\left(\mathcal{J}_1^\varepsilon - m_k\right)^2}\right)$$

which tends to zero by (3.7).

## 3.2 Fluctuations

**Proposition 3.2.** *If $P$ is a polynomial of degree $d$, then*

$$\left(\varepsilon^{-1/2}\int_0^t \left(P(\mathbf{S}_\varphi^\varepsilon(s)) - \tau\left(P(\mathbf{S}_\varphi^\varepsilon(s))\right)\right) ds\right)_{t\ge 0} \xrightarrow[\varepsilon\to 0]{(d)} \left(\sigma_S(P)\mathbf{S}(t)\right)_{t\ge 0} \tag{3.12}$$

*where*

$$\sigma_S(P)^2 = \sum_1^d c_q^2\sigma_q^2\,, \tag{3.13}$$

*with $P=\sum_{q=0}^d c_q T_q$ , $\sigma_q$ is defined in (2.7) and the convergence is fidi.*

We recall some facts about the convergence of free Wigner chaos associated to the free Brownian motion $(\mathbf{S}_t)_{t\geq 0}$ (we refer to Biane and Speicher [8] for stochastic calculus for free Brownian motion). For $f \in L^2(\mathbb{R}_+^n)$ with some symmetry, we can define a multiple stochastic integral of order $n$ with respect to $\mathbf{S}$, denoted by $I_n^S(f)$. In particular, for $f = 1_{[0,t]}^{\otimes n}$, $I_n^S(f) = T_n(\mathbf{S}_t)$ where $T_n$ is the $n$th Tchebycheff polynomial (the family of orthogonal polynomials for the semi circular distribution).
Let $(f_k)$ be a sequence of $L^2$ functions in $\mathbb{R}_+^n$. In [17], the authors proved that the convergence in distribution of a sequence $(I^S(f_k))_k$ when $k \to \infty$ to a semi circular distribution is equivalent to the convergence of the fourth moment to 2 (for a normalized sequence satisfying $||f_k||_2 = 1$) and is also equivalent to a condition expressed in terms of contractions of the $(f_k)$.
As a corollary, they obtain the following Wiener-Wigner transfert principle.

**Theorem 3.3.** *[17, Th. 1.8] Assume that the sequence $f_k$ is fully symmetric. Let $\sigma$ be a finite constant. Then, as $k \to \infty$,*

1. $\mathbb{E}[I_n^W(f_k)^2] \to n!\sigma^2$ *if and only if* $\mathbb{E}[I_n^S(f_k)^2] \to \sigma^2$ ,
2. *If the relations above are verified, then* $I_n^W(f_k) \longrightarrow^{(d)} \mathcal{N}(0; n!\sigma^2)$ *if and only if* $I_n^S(f_k) \longrightarrow^{(d)} \mathrm{SC}(0;\sigma^2)$ .

**Proof of Prop. 3.2** Let us first consider a fixed index $n$ and a fixed time $t = 1$. We want to prove the following result (analogous to the Breuer-Major theorem) :

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 T_n(\mathbf{S}_\varphi^\varepsilon(s))ds \xrightarrow[\varepsilon\to 0]{(d)} \mathrm{SC}(0;\sigma_n^2)\,. \tag{3.14}$$

By scaling property,

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 T_n(\mathbf{S}_\varphi^\varepsilon(s))ds \stackrel{(d)}{=} \sqrt{\varepsilon} \int_0^{\frac{1}{\varepsilon}} T_n(\mathbf{S}_\varphi^1(s))ds$$

and, with $\varepsilon^{-1} = T$, like (2.11)

$$T^{-1/2} \int_0^T T_n\left(\mathbf{S}_\varphi^1(t)\right) dt = I_n^S(f_T) \tag{3.15}$$

with $f_T$ defined in (2.10) and for $f \in L^2(\mathbb{R}_+^n)$

$$I_n^S(f) = \int_{\mathbb{R}_+^n} f(s_1, \dots, s_n) d\mathbf{S}_{s_1} \cdots d\mathbf{S}_{s_n}\,. \tag{3.16}$$

We can now apply the Wiener-Wigner transfer principle Theorem 3.3 to assert that from the convergence (2.13), we deduce the convergence in distribution (3.14).

Let us extend the above result to a polynomial $P(x) = \sum_1^d c_k T_k(x)$. We consider

$$Y_T = T^{-1/2} \int_0^T P(\mathbf{S}_\varphi^\varepsilon(s))ds = \sum_1^d c_k Y_{k,T}$$

where $Y_{k,T}$ is associated with $T_k$. We know that $Y_{k,T}$ converges in distribution to $\mathrm{SC}(0;\sigma_k^2)$. Moreover from the multi-dimensional Wiener-Wigner transfer theorem [20, Th. 1.6], the vector $(Y_{1,T},\ldots,Y_{d,T})$ converges in distribution to $(S_1,\ldots,S_d)$ where the $S_i$ are $\mathrm{SC}(0;\sigma_i^2)$ distributed and free since they are in distinct chaoses. To show that $Y_T$ converges to $\mathrm{SC}(0;\sum_1^d c_k^2\sigma_k^2)$ we have to consider the convergence of moments. Actually

$$(Y_T)^j = \left(\sum_1^d c_k Y_{k,T}\right)^j = Q(Y_{1,T},\ldots,Y_{d,T})$$

for some polynomial $Q$. From the above convergence, we deduce the convergence of $Q(Y_{1,T},\ldots,Y_{d,T})$ to $Q(S_1,\ldots,S_d) = (\sum_1^d c_k S_k)^j$. This means that $Y_T$ converges in distribution to $\mathrm{SC}(0;\sum_1^d c_k^2\sigma_k^2)$.

Let us fixed $n$. In view of proving the fidi convergence of the process

$$T^{-1/2} \int_0^{Tu} T_n\left(\mathbf{S}_\varphi^1(t)\right) dt \ , \ \ u \geq 0$$

to the free Brownian motion, we refer to Sec. 5.3. We consider four times $u_1 < u_2 < u_3 < u_4$ and the limit (2.14). Applying again he multi-dimensional Wiener-Wigner transfer theorem [20, Th. 1.6] we deduce that

$$\left(I_n^S(f_T^{[u_1,u_2]}), I_n^S(f_T^{[u_3,u_4]})\right) \xrightarrow[T\to\infty]{(d)} (S_1,S_2)$$

where $(S_1,S_2$ are free and distributed as $\mathrm{SC}(0;(u_2-u_1),\mathrm{SC}(0,;u_3-u_4))$.

An extension to a general polynomial and the fidi convergence can be obtained by a mixing of the above arguments. It is left to the reader. □

In particular for $\varphi = 1_{]-1,0]}$ we obtain

**Corollary 3.4.**

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 T_n\left(\frac{S(s+\varepsilon)-S(s)}{\sqrt{\varepsilon}}\right) ds \xrightarrow[\varepsilon\to 0]{(d)} \mathrm{SC}\left(0;\frac{2}{n+1}\right) . \tag{3.17}$$

# 4 Matricial case : LLN

In this section we consider the space $\mathcal{H}_N$ of $N\times N$ Hermitian matrices and we denote by tr the normalized trace $N^{-1}\mathrm{Tr}$. We replace the bilateral Brownian motion of Sec. 2 by a bilateral Hermitian Brownian motion $W^{(N)}$, with values in the space $\mathcal{H}_N$ such that $(W^{(N)}_{i,j}, i \leq j)$ are independent bilateral Brownian motions, complex if $i<j$ and real if $i=j$ with variance

$$\mathbb{E}|W^{(N)}_{i,j}(t)|^2 = |t|/N\,.$$

Set

$$\mathbb{W}^{\varepsilon,N}_{\varphi}(t) = \varepsilon^{-1/2}\int_{-\infty}^{\infty}\varphi\left(\frac{t-s}{\varepsilon}\right)dW^{(N)}(s)\,,$$

with the usual assumption: $\varphi$ compactly supported and satisfying $\|\varphi\|_2 = 1$. For $t=1$, the distribution is the classical Gaussian Unitary Ensemble of size $N$ and variance $N^{-1}$ which we will denote by GUE($N^{-1}$) in the sequel. We denote by $M^{(N)}$ a random matrix distributed as GUE($N^{-1}$). We will often omit the index $\varphi$ when it will be clear.

**Proposition 4.1.** *Assume that $N$ is fixed. Then, almost surely*

$$\forall k \geq 1\ ,\ \lim_{\varepsilon}\int_0^1 \left(\mathbb{W}^{\varepsilon,N}(t)\right)^k dt = \mathbb{E}(M^{(N)})^k\,. \tag{4.1}$$

The proof is componentwise. We have

$$\left[\int_0^1 \left(\mathbb{W}^{\varepsilon,N}(t)\right)^k dt\right]_{i,j} = \sum_{i_2,\dots,i_{k-1}}\int_0^1 \left(\mathbb{W}^{\varepsilon,N}(t)_{i,i_2}\dots\mathbb{W}^{\varepsilon,N}(t)_{i_{k-1},j}\right)dt\,,$$

Each integrand is selfsimilar, $2a$-dependent and stationary as in Sec. 1, so

$$\lim_{\varepsilon}\int_0^1 \left(\mathbb{W}^{\varepsilon,N}(t)_{i,i_2}\dots\mathbb{W}^{\varepsilon,N}(t)_{i_{k-1},j}\right)dt = \mathbb{E}\left(M^{(N)}_{i,i_2}\dots M^{(N)}_{i_{k-1},j}\right) \tag{4.2}$$

a.s. and we get (4.1) by summation. □

Let us now consider the asymptotics $N\to\infty$.

In all the sequel, $P$ will be a polynomial in one variable. Let us consider the following diagram

$$\begin{array}{ccc}
\int_0^1 P\left(\mathbb{W}^{\varepsilon,N}(s)\right) ds \in \mathcal{H}_N & \xrightarrow{(1):\varepsilon\to 0} & \mathbb{E}\left(P(M^{(N)})\right) \in \mathcal{H}_N \\
{\scriptstyle (3):N\to\infty}\big\downarrow & & {\scriptstyle (2):N\to\infty}\big\downarrow \\
\int_0^1 P\left(\mathbf{S}^{\varepsilon}(s)\right) ds \in \mathcal{A} & \xrightarrow{(4):\varepsilon\to 0} & \int_0^1 P(x) d\,\mathrm{SC}(x)\cdot \mathbf{1} \in \mathcal{A}\,.
\end{array} \tag{4.3}$$

Let us give the precise meaning of all these arrows, at least when $P$ is a monomial $P(x) = x^k$. (1) is the result of Prop. 4.1 and (4) was proved in Sec. 3.1.

Let us prove (2). Set $H_N := \mathbb{E}\left(P(M^{(N)})\right)$ and $c := \int_0^1 P(x)d\,\mathrm{SC}(x)$. (2) means that

$$\lim_{N\to\infty} \mathrm{tr} H_N^j = c^j, \text{ for all } j\,. \tag{4.4}$$

Actually,

$$H_N^j = \mathbb{E}\left[P(M_{(1)}^{(N)})\cdots P(M_{(j)}^{(N)})\right]$$

where $M_{(1)}^{(N)}, \ldots, M_{(j)}^{(N)}$ are i.i.d. GUE$(N^{-1})$. From the Wigner's theorem, we have then

$$\lim_{N\to\infty} \mathrm{tr} H_N^j = \tau\left[P(S_1)\cdots P(S_j)\right]\,,$$

where $S_1, \ldots, S_j$ are free and SC. The RHS is clearly $c^j$.

Let us prove (3). Assume $\varepsilon = 1$ to simplify notations. Set

$$J_j^N = \mathrm{tr}\left(\int_0^1 \mathbb{W}^{1,N}(s)^k ds\right)^j\,.$$

We have to prove that

$$\lim_N \mathbb{E} J_j^N = \tau\left(\int_0^1 (\mathbf{S}^1(s)^k ds\right)^j \tag{4.5}$$

and, if possible

$$\lim_N J_N^j = \tau\left(\int_0^1 (\mathbf{S}^1(s)^k ds\right)^j \text{ (a.s.)}\,. \tag{4.6}$$

<u>Let us begin with $j = 1$.</u>

Since $\mathbb{W}^{1,N}(s) \overset{(d)}{=} M^{(N)}$ and $\mathbf{S}^1(s) \overset{(d)}{=}$ SC, we have

$$\mathbb{E} J_1^N = \mathrm{tr} \int_0^1 \mathbb{E}\left(\mathbb{W}^{1,N}(s)^k\right) = \mathrm{tr}\ \mathbb{E}(M^{(N)})^k$$

$$\tau\left(\int_0^1 \mathbf{S}^1(s)^k\right) ds = \int_0^1 \tau\left(\mathbf{S}^1(s)^k\right) ds = m_k\,, \tag{4.7}$$

hence from the convergence (2)

$$\lim_N \mathbb{E} J_1^N = \lim_N \mathbb{E}\mathrm{tr}\left(M^{(N)}\right)^k = m_k = \tau\left(\int_0^1 \left(\mathcal{S}^1(s)\right)^k\right) ds\,. \tag{4.8}$$

To prove the a.s. convergence for $j = 1$, we will use the second moment method. Let us know look for $\mathrm{Var} J_1^N$. From the classical Fubini-like formula

$$\mathrm{Var}\left(\int_0^1 T(s)ds\right) = \int_{[0,1]^2} \mathrm{Cov}(T(s), T(u))dsdu\,,$$

where $T$ is any $L^2$ process, we get

$$\mathrm{Var} J_1^N = \int_{[0,1]^2} \mathrm{Cov}\left(\mathrm{tr}\mathbb{W}^{1,N}(s)^k, \mathrm{tr}\mathbb{W}^{1,N}(u)^k\right) dsdu\,.$$

Now, by Cauchy inequality and stationarity

$$\left|\mathrm{Cov}\left(\mathrm{tr}\mathbb{W}^{1,N}(s)^k, \mathrm{tr}\mathbb{W}^{1,N}(u)^k\right)\right| \leq \mathrm{Var}\left(\mathrm{tr}(M^{(N)})^k\right)\,, \tag{4.9}$$

which tends to 0 as $N \to 0$ (see [2, Sec. 2.1.4]. Actually, since the bound is $O(N^{-2})$ the convergence may be strenghtened into an a.s. convergence, owing to Borel-Cantelli lemma.

For $j \geq 2$, let us rewrite $J_j^N$ as a multiple integral

$$J_j^N = \mathrm{tr}\int_{[0,1]^j} \mathbb{W}^{1,N}(s_1)^k \cdots \mathbb{W}^{1,N}(s_j)^k ds_1 \cdots ds_j\,, \tag{4.10}$$

so that, by Fubini

$$\mathbb{E} J_j^N = \int_{[0,1]^j} \mathbb{E}\mathrm{tr}\left(\mathbb{W}^{1,N}(s_1)^k \cdots \mathbb{W}^{1,N}(s_j)^k\right) ds_1 \cdots ds_j\,, \tag{4.11}$$

It should be clear that the process $\mathbb{W}^{1,N}$ converges to $\mathbf{S}^1$ in the sense that for every sequence $(s_1, \ldots, s_p)$

$$\lim_N \mathbb{E}\mathrm{tr}[\mathbb{W}^{1,N}(t_1)\cdots\mathbb{W}^{1,N}(t_p)] = \tau\left(\mathbf{S}^1(t_1)\cdots\mathbf{S}^1(t_p)\right)\,.$$

So, we have a convergence pointwise of the integrand in (4.11) to

$$\tau\left(\mathbf{S}^1(s_1)^k \cdots \mathbf{S}^1(s_j)^k\right)\,.$$

Up to an application of the Lebesgue dominated theorem, we could conclude (again by Fubini)

$$\lim_N \mathbb{E}J_j^N = \int_{[0,1]^j} \tau\left(\mathbf{S}^1(s_1)^k \cdots \mathbf{S}^1(s_j)^k\right) ds_1 \cdots ds_j$$
$$= \tau\left(\int_{[0,1]^j} \mathcal{S}^1(s_1)^k \cdots \mathbf{S}^1(s_j)^k ds_1 \cdots ds_j\right) = \tau\left(\int_{[0,1]} \mathbf{S}^1(s)^k ds\right)^j .$$

Now, we have to find a uniform bound for $\mathbb{E}\mathrm{tr}\left(\mathbb{W}^{1,N}(s_1)^k \cdots \mathbb{W}^{1,N}(s_j)^k\right)$. By Holder inequality

$$|\mathbb{E}\mathrm{tr}(A_1 \dots A_j)| \leq \prod_{r=1}^{j} \left(\mathbb{E}\mathrm{tr}(A_r)^j\right)^{1/j} ,$$

so that

$$|\mathbb{E}\mathrm{tr}\left(\mathbb{W}^{1,N}(s_1)^k \cdots \mathbb{W}^{1,N}(s_j)^k\right)| \leq \prod_1^j \left(\mathbb{E}\mathrm{tr}(\mathbb{W}^{1,N}(s_r)^{kj}\right)^{1/j} = \mathbb{E}\mathrm{tr}(M_N)^{kj}$$

which is bounded in $N$ since it converges to $\tau\left((\mathbf{S}^1)^{kj}\right)$.

# 5 Matricial case : fluctuations

As seen in the above sections, the Hermite polynomials in the scalar case and the Tchebycheff polynomials in the free case play a major role for the study of fluctuations. In the matricial case, we will use Hermite trace polynomials which are matrix valued of matrix variate. They are defined in the next subsection. This can be considered as an intermediate link between the scalar and free cases.

Besides, it is also possible to consider polynomials which are scalar valued of matrix variate. They are studied in Sec. 6.

## 5.1 Hermite polynomials for Hermitian Brownian motion

This section is a summary of the paper of Anshelevich and Buzinski [3] in which the authors define the Hermite trace polynomials for the Hermitian Brownian motion.

First, let us recall what is known for the real Brownian motion $(B_t)_{t\geq 0}$. There exists polynomials $H_n(x,t)$ such that for all $n$, $H_n(B_t,t)$ is a martingale for the filtration induced by $B$. More precisely, $H_n(x,t) = t^{n/2}H_n(\frac{x}{\sqrt{t}})$

where $H_n$ is the classical Hermite polynomial of degree $n$. Moreover, we have the chaos representation

$$H_n(B_t, t) = \int_{[0,t]^n} dB_{t_1} \dots dB_{t_n}.$$

Let $(W^{(N)}(t))$ be a $N \times N$ Hermtian Brownian motion. We cannot find a polynomial $P(x,t)$ of degree $n$ in $x$ such that $P(W^{(N)}(t), t)$ is a martingale, for $n \geq 3$. For example, for $n = 3$, a martingale $M^{(N)}$ involving $(W^{(N)}(t))^3$ is given by :

$$M^{(N)}(t) = (W^{(N)}(t))^3 - t(2W^{(N)}(t) + \operatorname{tr}(W^{(N)}(t))). \tag{5.1}$$

Note that for $N = 1$, we recover the classical Hermite polynomial $H_3(x,t) = x^3 - 3tx$.

Therefore, we need to replace the class of polynomial by a larger class of trace polynomials. Informally, a trace polynomial in a matrix indeterminate $X$ is a linear combination of product of the form $X^k \prod_i \operatorname{tr}(X^{l_i})$.

From now in this section, we denote by $W$ a $N \times N$ Hermitian Brownian motion as $W^{(N)}$ in Sec. 4 but we drop the superscript $N$ for simplicity. For $h \in L^2(\mathbb{R})$, we set

$$W(h) = \int_{-\infty}^{\infty} h(t) dW(t).$$

In [3], the authors define trace polynomials in GUE matrices. A trace polynomial in $W(h_1) \dots, W(h_n)$ is indexed by $\alpha \in S_0(n)$, where $S_0(n)$ denotes the set of permutations of $[0,n] := \{0, 1, \dots, n\}$ and is defined by

$$\operatorname{Tr}_\alpha(W(h_1) \dots, W(h_n)) = \prod_{i \text{ in the cycle starting with } 0} W(h_i) \prod_{\text{others cycles}} \operatorname{Tr}\left( \prod_{i \text{ in the cycle}} W(h_i) \right). \tag{5.2}$$

The matricial $\alpha$-Hermite polynomial $H_\alpha(W_t)$ is defined as linear combinations of trace polynomial $\operatorname{Tr}_\eta(W_t, \dots, W_t)$, where $\eta$ is a contraction $C_\pi(\alpha)$ of $\alpha$. See [3, Def. 3.13, Cor. 6.4]. We give some examples of Hermite polynomials in Prop. 5.1.
We now precise the definition of the contractions.

**Definition.** (see [3, Definition 3.9] Let $\pi$ be a partition of $[n] := \{1, \dots, n\}$ with blocks of size 1 or 2, i.e. $\pi \in \mathcal{P}_{1,2}(n)$ and denote $\operatorname{supp}(\pi) = [n] \backslash \operatorname{Sing}(\pi)$

the set of the 2-blocks. For $\alpha \in S_0(n)$ and $\pi \in \mathcal{P}_{1,2}(n)$, we define

$$
\begin{aligned}
C_\pi(\alpha) &= \left(\frac{1}{N}\right)^{a(\alpha,\pi)} \beta_\pi(\alpha) \\
\beta_\pi(\alpha) &= P_{[0,n-2l]}^{[0,n]\setminus \mathrm{supp}(\pi)}(\pi\alpha)\big|_{\mathrm{supp}(\pi)^c}\,,
\end{aligned} \tag{5.3}
$$

where

1. $l$ is the number of 2-blocks of $\pi$.
2. For $A, B$ two sets of integers with the same cardinal, $P_B^A$ denotes the unique order-preserving bijection from $A$ to $B$ and the corresponding bijection on the set of permutations on $A$, resp. on $B$. [1]
3. $a(\alpha,\pi) = \mathrm{cyc}_0((\pi\alpha)|_{\mathrm{supp}(\pi)^c} - \mathrm{cyc}_0(\pi\alpha) + l$ and $\mathrm{cyc}_0$ denotes the number of cycles of a permutation on $[0,n]$, not containing $0$.

For $\alpha \in S_0(n), u \in \mathbb{R}$ and $M \in \mathcal{H}_N$, set

$$
\widetilde{H}_\alpha(M,u) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{n-|\pi|} \left(\frac{1}{N}\right)^{a(\alpha,\pi)} u^l \mathrm{Tr}_{\beta_\pi(\alpha)}(M,\ldots,M) \tag{5.4}
$$

where in the last Tr there are $n-2l$ arguments.
The $\alpha$ Hermite polynomial is defined by

$$
H_\alpha(W(h)) = \widetilde{H}_\alpha(W(h), \|h\|^2)\,, \tag{5.5}
$$

*Remark* 2. Note that if $\pi$ has only 2-cycles, $C_\pi(\alpha)$ is equal to (0) up to multiplicative constant.

Some examples are in the following proposition

**Proposition 5.1.** *Let $\alpha_n = (01\ldots n)$. The first $\widetilde{H}_{\alpha_n}$ are :*

$$
\begin{aligned}
&\widetilde{H}_{\alpha_1}(M,u) = -M\,,\\
&\widetilde{H}_{\alpha_2}(M,u = M^2 - u\,,\\
&\widetilde{H}_{\alpha_3}(M,u) = -M^3 + 2uM + u\mathrm{tr}M\,,\\
&\widetilde{H}_{\alpha_4}(M,u) = M^4 - 2uM\mathrm{tr}M - 3uM^2 - u\mathrm{tr}M^2 + (2+N^{-2})u^2\,.
\end{aligned}
$$

Let us notice that for $N=1$ we recover the classical Hermite polynomials, except for the sign. This comes from the definition (5.5) which, in [3], obeys algebraic motivations.

[1]Example ([3, Ex. 3.8]): For $\alpha = (13524)$ and $S = \{2,5\}$, $\alpha|_{S^c} = (134)$ and $P_{[3]}^{[5]\setminus S}\alpha|_{S^c} = (123)$.

## 5.2 Proof of Proposition 5.1

In the following we set $q = 1/N$, $c = a(\alpha, \pi)$ and $\beta = P_{[0,n-2l]}^{[0,n]\setminus\mathrm{supp}(\pi)}(\pi\alpha)\big|_{\mathrm{supp}(\pi)^c}$.

For $n = 1$, $\mathcal{P}_{1,2}(n) = (1), l = 0, n - |\pi| = 1, \pi\alpha = (01), \beta = (01), \mathrm{Tr}_\beta = M, q^c = 1$.

For $n = 2$ we have

| $\pi$ | $2-\lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 2 | (012) | (012) | $M^2$ | 0 | 1 |
| (12) | 1 | (02)(1) | (0) | 1 | 1 | 1 |

For $n = 3$

| $\pi$ | $3-\lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 3 | (0123) | (0123) | $M^3$ | 0 | 1 |
| (1)(23) | 2 | (013)(2) | (01) | $M$ | 1 | 1 |
| (2)(13) | 2 | (03)(12) | (0)(1) | $\mathrm{Tr}\, M$ | 1 | $N^{-1}$ |
| (12)(3) | 2 | (023)(1) | (01) | $M$ | 1 | 1 |

For $n = 4$

| $\pi$ | $4-\lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 4 | (01234) | (01234) | $M^4$ | 0 | 1 |
| (1)(23)(4) | 3 | (0134)(2) | (012) | $M^2$ | 1 | 1 |
| (1)(2)(34) | 3 | (0124)(3) | (012) | $M^2$ | 1 | 1 |
| (1)(24)(3) | 3 | (014)(23) | (01)(2) | $M\mathrm{Tr}\, M$ | 1 | $N^{-1}$ |
| (14)(2)(3) | 3 | (04)(123) | (0)(12) | $\mathrm{Tr}\, M^2$ | 1 | $N^{-1}$ |
| (13)(2)(4) | 3 | (034)(12) | (02)(1) | $M\mathrm{Tr}\, M$ | 1 | $N^{-1}$ |
| (12)(3)(4) | 3 | (0234)(1) | (012) | $M^2$ | 1 | 1 |
| (12)(34) | 2 | (024)(1)(3) | (0) | 1 | 2 | 1 |
| (13)(24) | 2 | (03214) | (0) | 1 | 2 | $N^{-2}$ |
| (14)(23) | 2 | (04)(13)(2) | (0) | 1 | 2 | 1 |

We now give some moment formulas for tracial polynomials of Hermitian Brownian and for products of Hermite polynomials :

**Proposition 5.2.** *See [3, Prop. 6.1] Let $\{D_1, \dots D_n\}$ be non random Hermitian matrices. Then for even $n$, for $\alpha \in S_0(n)$,*

$$\mathbb{E}(\mathrm{Tr}_\alpha(W(h_1)D_1, \dots, W(h_n)D_n)) = \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{P}_2(n)} C_\pi(h_1 \otimes \dots \otimes h_n)\mathrm{Tr}_{\pi\alpha}(D_1, \dots, D_n). \tag{5.6}$$

*where $\mathcal{P}_2(n)$ is the set of pair partitions of $[n]$ and*

$$C_\pi(h_1 \otimes \ldots \otimes h_n) = \prod_{(i,j) \text{ pair of } \pi} \langle h_i, h_j \rangle.$$

**Proposition 5.3.** *See [3, Prop. 5.15]. For $\alpha_n = (01\ldots n)$, $k \in \mathbb{N}$,*

$$\mathbb{E}(H_{\alpha_n}(W(h_1))\ldots H_{\alpha_n}(W(h_k))) = \sum_{\pi \in \mathcal{P}_2(n,\ldots,n)} C_\pi(\alpha_{nk}) \quad C_\pi(h_1^{\otimes n} \otimes \ldots \otimes h_k^{\otimes n}) = \sum_{\pi \in \mathcal{P}_2(n,\ldots,n)} \left(\frac{1}{N}\right)^{nk/2 - cyc_0(\pi\alpha)} \quad C_\pi(h_1^{\otimes n} \otimes \ldots \otimes h_k^{\otimes n}). \tag{5.7}$$

*where $\mathcal{P}_2(n,\ldots,n)$ is the set of inhomogeneous pair partitions of $[nk]$, meaning that if $(i,j)$ is a pairing of the partition with $i < j$, then $i \leq pn < j$ for some $1 \leq p \leq k-1$ (i.e. considering the set $nk$ as $k$ blocks of $n$ elements, any pairing involves two elements of different blocks).*
*In (5.7), with a slight abuse of notation, $C_\pi(\alpha_{nk})$ denotes the constant $K$ in the writing of the partition $C_\pi(\alpha_{nk}) = K(0)$ and the LHS is equal to the RHS multiplied by the identity matrix.*

### 5.3 Fluctuations

Let $(W_t,\ t \in (-\infty,\infty))$ be as above. Let $\varphi : \mathbb{R} \to \mathbb{R}$ compactly supported with bounded variation and satisfying $\|\varphi\|_2 = 1$. Set $\varphi_t(s) = \varphi(t-s)$. We now consider the stationary process with value in $\mathcal{H}_N$ of $N \times N$ defined by :

$$X_t = \int_{-\infty}^{\infty} \varphi(t-s) dW_s := \int_{-\infty}^{\infty} \varphi_t(s) dW_s = W(\varphi_t)$$

Notice that

$$\langle h_t, h_s \rangle = \rho(t-s) \tag{5.8}$$

where $\rho(t) = \int \varphi(t+u)\varphi(u)du$ (2.5).
Let $H_{\alpha_n}$ the Hermite polynomial on $\mathcal{H}_N$ with $\alpha_n = (01\ldots n) \in S_0(n)$.
As in the scalar case, we are looking for a CLT for the random matrix :

$$M_T = \frac{1}{\sqrt{T}} \int_0^T H_{\alpha_n}(X_s) ds$$

as $T$ tends to $\infty$. The main result of this section is :

**Theorem 5.4.** *As $T \to \infty$, $M_T$ converges in distribution to a Gaussian random matrix $M_\infty$ whose law is characterized by the following:*

1) *It is invariant by conjugation which implies that in the decompostion $M_\infty = U^* D U$, the matrix $U$ of eigenvectors is Haar distributed and independent of the matrix $D$ of eigenvalues.*

2) *The law of the spectral distribution*

$$\mu_{M_\infty} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(M_\infty)}$$

*is given by its moments : for $k$ even,*

$$\mathbb{E}\left( \int x^k d\mu_{M_\infty}(x) \right) = \mathbb{E}(\mathrm{tr}(M_\infty^k)) = K_{k,n,N} \sigma_n^k \tag{5.9}$$

*where $\sigma^2$ is defined in (2.7),*

- $K_{k,n,N} = \sum_{\pi \in \Pi_{n,k}} C_\pi(\alpha_{nk})$ *where $C_\pi$ is defined in (5.3)*
- $\Pi_{n,k}$ *is the set of partitions $\pi$ of $\mathcal{P}_2(n, \dots, n)$ in inhomogeneous pairs such that if $i_1$ and $i_2$ are in the same block of size $n$, then $\pi(i_1)$ and $\pi(i_2)$ are also in the same block.*

3) *We have*

$$M_\infty \overset{(d)}{=} \sigma_n \left( a_{n,N} G + b_{n,N} \xi I_N \right) \tag{5.10}$$

*where*

- $G \overset{(d)}{=} \mathrm{GUE}(N^{-1})$, $\xi \overset{(d)}{=} \mathcal{N}(0; N^{-2})$ *(real) and $G$ and $\xi$ are independent,*
- *the coefficients $a_{n,N}$ and $b_{n,N}$ are defined via the permutation $\tilde{\alpha} = (0)(1 \dots n)(n+1 \dots 2n)$ by*

$$a_{n,N}^2 = \frac{1}{N^n} \sum_{\pi \in \mathcal{P}_2'(n,n)} N^{\mathrm{cyc}(\pi\tilde{\alpha})} \tag{5.11}$$

$$b_{n,N}^2 = \frac{1}{N^n} \sum_{\pi \in \mathcal{P}_2(n,n) \setminus \mathcal{P}_2'(n,n)} N^{\mathrm{cyc}(\pi\tilde{\alpha})} \tag{5.12}$$

*with*

$$\mathcal{P}_2'(n,n) = \{\pi \in \mathcal{P}_2(n,n) : n \text{ and } 2n \text{ are in the same cycle of } \pi\tilde{\alpha}\}.$$

*Remark* 3. It could seem natural to tackle this problem with the tools of Wiener chaos, as in the scalar case. Unfortunately, we did not find any ready-made version of chaos for matrix-valued Brownian motion. It is remarkable that in [17, Remark 1.40], the authors leave to the reader this "involved" task.

**Proof:** The convergence of $M_T$ to a Gaussian random matrix follows from Breuer-Major CLT theorem for non-linear functional of stationary Gaussian process (extended to multidimensional Gaussian process by Arcones). Indeed, for any non random Hermitian matrix $A$, $\mathrm{Tr}\,(AM_T)$ can be written as $\frac{1}{\sqrt{T}}\int_0^T F(X_s)ds$ for some function $F$ from $\mathcal{H}_N$ to $\mathbb{R}$, centered for the multidimensional stationary Gaussian process $(X_t)$, and thus converges to a centered Gaussian variable of variance $\sigma^2(A,n,N)$. It follows that $M_T$ converges to a Gaussian Hermitian matrix.

1) Since the distribution of $(X_t)_{t\geq 0}$ is invariant by unitary conjugation, it follows that for all unitary matrix $U$, $M_T \overset{(d)}{=} UM_TU^*$ and therefore $M_\infty \overset{(d)}{=} UM_\infty U^*$. We deduce (see [2, Cor. 2.5.4]) that the eigenvectors of $M_\infty$ are Haar distributed and are independent of the eigenvalues.

2) Let us compute for $k \geq 1$ the moment

$$\int x^k d\mu_{M_\infty}(x) = \mathbb{E}(\mathrm{tr}(M_\infty^k))\,,$$

and prove (5.9).

We have

$$M_T^k = T^{-k/2}\int_{[0,T]^k} H_\alpha(X_{t_1})H_\alpha(X_{t_2})\dots H_\alpha(X_{t_k})dt_1\cdots dt_k\,. \tag{5.13}$$

From (5.7),

$$\mathbb{E}M_T^k = \sum_{\pi\in\mathcal{P}_2(n,\dots,n)} C_\pi(\alpha_{nk})\; T^{-k/2}\int_{[0,T]^k} C_\pi(h_{t_1}^{\otimes n}\otimes\dots\otimes h_{t_k}^{\otimes n})dt_1\cdots dt_k\,. \tag{5.14}$$

From (5.8) we have

$$C_\pi\left(h_{t_1}^{\otimes n}\otimes\cdots\otimes h_{t_k}^{\otimes n}\right) = \prod_{1\leq i<j\leq k}\rho(t_i-t_j)^{n_{ij}} \tag{5.15}$$

where $n_{ij}$ denotes the number of pairs consisting in an element of the block $i$ and an element of block $j$. Of course $0\leq n_{ij}\leq n$. If $n_{ij}=n$ the blocks

$i$ and $j$ are completely connected. Let us denote by $J$ those pairs of blocks such that $n_{ij} = n$ and let $|J|$ be its cardinal. We can split

$$\int_{[0,T]^k} C_\pi \left( h_{t_1}^{\otimes n} \otimes \cdots \otimes h_{t_k}^{\otimes n} \right) dt_1 \dots dt_k =$$
$$\int \prod_{(i,j)\in J} \rho(t_i - t_j)^{n_{ij}} 1_{[0,T]}(t_i) dt_i 1_{[0,T]}(t_j) dt_j$$
$$\times \int \prod_{(i,j)\notin J} \rho(t_i - t_j)^{n_{ij}} \prod_{s=1}^{k-2|J|} 1_{[0,T]}(t_s) dt_s \,. \tag{5.16}$$

The first factor is actually

$$\left( \int_{[0,T]^2} \rho(t_2 - t_1)^n dt_1 dt_2 \right)^{|J|} = \left( \int_{[-T,T]} \rho(t)^n (T - |t|) dt \right)^{|J|} , \tag{5.17}$$

which implies

$$\lim_{T\to\infty} T^{-|J|} \left( \int_{[0,T]^2} \rho(t_2 - t_1)^n dt_1 dt_2 \right)^{|J|} = \left( \int \rho(t)^n dt \right)^{|J|} . \tag{5.18}$$

In the second factor, if the complement of $J$ is not empty, we denote by $r = k - 2|J|$ be the number of remaining blocks.

We can consider a graph $G$ whose vertices are the $r$ variables of integration and whose edges are the pairs $(k, \ell)$ such that $n_{k\ell} \neq 0$. Each vertex has a degree greater or equal to 2, which implies that the number of edges is greater or equal to $r$. If $G$ is connected, we consider a covering tree. It has $r - 1$ edges, and each one can be identified to a linear form

$$(i, j) \sim (t_1, \dots, t_r) \mapsto t_i - t_j \,.$$

If these forms are denoted by $\ell_1, \dots, \ell_{r-1}$, the kernel of linear map

$$(t_1, \dots, t_r) \mapsto (\ell_1, \dots, \ell_{r-1})$$

is the diagonal $t_1 = \cdots = t_r$, hence the rank of the linear map is $r - 1$. Adding one coordinate, say $t_r$, we can perform the change of variable

$$(t_1, \dots, t_r) \mapsto (u_1 = \ell_1, \dots, u_{r-1} = \ell_{r-1}, u_r = t_r) \,.$$

Since its Jacobian is 1 we get

$$\int \prod_{(i,j)\notin J} \rho(t_i - t_j)^{n_{ij}} \prod_{k=1}^{r} 1_{[0,T]}(t_k) dt_s \leq \int \prod_{(i,j)\notin J} \rho(t_i - t_j) \prod_{s=1}^{r} 1_{[0,T]}(t_s) dt_s$$
$$\leq \int_{\mathbb{R}^{r-1}\times[0,T]} \left( \prod_{s=1}^{r-1} \rho(u_s) du_s \right) du_r \leq T \left( \int \rho(t) dt \right)^{r-1} .$$

Now, if $G$ has $c > 1$ connected components, we cinder each one separately and we conclude

$$\int \prod_{(i,j)\notin J} \rho(t_i - t_j)^{n_{ij}} \prod_{s=1}^{r} 1_{[0,T]}(t_k) dt_s \leq T^c \left( \int \rho(t) dt \right)^{r-c} . \tag{5.19}$$

Actually , since each component has at least 3 vertices, we have $3c \leq r$ and we conclude

- If $|J| = k/2$ then

$$\lim_{T} T^{-k/2} \int_{[0,T]^k} C_\pi \left( h_{t_1}^{\otimes n} \otimes \cdots \otimes h_{t_k}^{\otimes n} \right) dt_1 \dots dt_k = \left( \int \rho(t)^n dt \right)^{k/2} \tag{5.20}$$

- If $|J| < k/2$, then

$$T^{-k/2} \int_{[0,T]^k} C_\pi \left( h_{t_1}^{\otimes n} \otimes \cdots \otimes h_{t_k}^{\otimes n} \right) dt_1 \dots dt_k =$$
$$O(T^{c-k/2+|J|}) = O(T^{c-r/2}) = O(T^{-r/6}) . \tag{5.21}$$

Therefore, the only pair partitions giving a non zero term as $T \to +\infty$ are the inhomogeneous partitions for which $n_{ij} = n$ for all $i \neq j$, thus a partition of $\Pi_{n,k}$.

3) It is known that a random symmetric isotropic real Gaussian matrix is of the form $aG + +b\eta I_N$ with

- $G \overset{(d)}{=} \mathrm{GOE}(N^{-1})$,
- $\eta \overset{(d)}{=} \mathcal{N}(0;1)$ independent of $G$,
- $a, b$ real,

see for instance [11, Lemma 4] or [10, Sec. 2.1]. The proof is easily extended to the Hermitian case with GUE instead of GOE. Moreover, owing to the expression of moments (5.9), it is clear that the coefficients $a$ and $b$ are proportional to $\sigma_n$, so that $a_{n,N}$ and $b_{n,N}$ do not depend on $\rho$ and are purely combinatorial.

To characterize $a_{n,N}$ and $b_{n,N}$, it is natural to compute the distribution of $\mathrm{Tr}\, AM_\infty$ for $A$ a Hermitian test matrix. On the one hand it is Gaussian with variance

$$\sigma^2_{n,N}(A) = \sigma^2_n \left(a^2_{n,N}\mathrm{tr}(A^2) + b^2_{n,N}(\mathrm{tr}(A))^2\right) . \tag{5.22}$$

On the other hand $\mathrm{Tr}\, AM_\infty$ as the limit of the Gaussian stationary process $M_T$ has a variance

$$\sigma^2_{n,N}(A) = 2\int_0^\infty \rho_{A,n,N}(t)dt \tag{5.23}$$

$$\rho_{A,n,N}(t) = \mathbb{E}\left(\mathrm{Tr}\left(AH_\alpha(X_t)\mathrm{Tr}\left(AH_\alpha(X_0)\right)\right.\right. . \tag{5.24}$$

A careful computation could make appear the terms $\mathrm{tr}(A^2)$ and $(\mathrm{tr}A)^2$. From the definition (5.5) of Hermite polynomials as linear combination of tracial monomials of Hermitian BM and formula (5.6) for the moments of tracial monomials, we can deduce that $\rho_{A,n,N}(t)$, as a function of $A$, is a linear combination (depending on $n, N, t$) of terms of the form $\mathrm{Tr}_\beta(D_1, \ldots, D_{2n})$ for $\beta \in S_0(2n)$ and the matrices $D_i$ are equal to $\mathrm{Id}_N$ except for two indexes where $D = A$. It follows that $\rho_{A,n,N}(t)$ is a linear combination of $(\mathrm{Tr}\, A)^2$ and $\mathrm{Tr}\,(A^2)$. Moreover, the coefficients of this combination must be proportional to $\rho(t)^n$. In the expansion of $\rho_{A,n,N}(t)$ we have to isolate the terms proportional to $\mathrm{tr}(A^2)\rho^n(t)$ and $(\mathrm{tr}A)^2\rho^n(t)$, respectively.

If we plug the expansions (5.5) of $H_\alpha(X_t)$ and $H_\alpha(X_0)$ in (5.24) we see that $\rho_{A,n,N}(t)$ is as um of terms like

$$\mathbb{E}\mathrm{Tr}\left[A\mathrm{Tr}_{\,\beta_{\pi(\alpha)}}(W(h), \ldots, W(h))A\mathrm{Tr}_{\,\beta_{\pi'(\alpha)}}(W(h_0)\ldots W(h_0))\right]$$

and using the definition (5.2) this can be written as

$$\mathbb{E}\left[\mathrm{Tr}\left[AW(h)^u AW(h_0)^{u'}\prod_i \mathrm{Tr}\,[W(h)^{v_i}]\prod_j \mathrm{Tr}\left[W(h_0)^{v'_j}\right]\right]\right] ,$$

for convenient $u, u', v_i, v_j$. Now, owing to (5.2) again, it is of the form

$$\mathbb{E}\mathrm{Tr}_{\,\tilde\alpha}\,(X, \ldots, X, XA, Y, \ldots, Y, YA, X, \ldots, X, Y, \ldots, Y) \tag{5.25}$$

where

- $X = W(h), Y = W(h_0)$,
- there are $r-1$ successive $X$, then $XA$, then $s-1$ successive $Y$ then $YA$ ...
- $$\tilde{\alpha} = (0)(1 \dots r)(r+1, \dots, r+s)\gamma\gamma'$$

  with $\gamma$ acting on the last $X$'s and $\gamma'$ acting on the last $Y$'s.

If we now apply the formula (5.6) we see that the occurence of $\rho(t)$ comes from the contractions $C_\pi$. If we want to get $\rho(t)^n$, the number of variables needs to be $2n$, $\pi$ must have no singleton, and also $r = s = n$ and then $\gamma = \gamma' = \emptyset$. Eventually

$$\tilde{\alpha} = (0)(1 \dots n)(n+1, \dots, 2n) \ , \pi \in \mathcal{P}_2(n,n) \, .$$

Under this condition, for the term $\mathrm{Tr}_{\pi\tilde{\alpha}}[(I_N, \dots, I_N, A, I_N, \dots, I_N, A)$ of (5.6) there two cases:

- if $n$ and $2n$ are in the same cycle of $\pi\tilde{\alpha}$ and we get a term in $\mathrm{Tr}\,(A^2)$ if with coefficient $N^{\mathrm{cyc}(\pi\tilde{\alpha})-1}$ i.e. a term in $\mathrm{tr}(A^2)$ with coefficient $N^{\mathrm{cyc}(\pi\tilde{\alpha})}$
- ortherwise we get a term in $(\mathrm{Tr}\,A)^2$ with coefficient $N^{\mathrm{cyc}(\pi\tilde{\alpha}-2}$ i.e. a term in $(\mathrm{tr}A)^2$ with coefficient $N^{\mathrm{cyc}(\pi\tilde{\alpha})}$.

A comparison with (5.22) ends the proof. □.

### 5.4 Example

We will give the values of $a_{n,N}$ and $b_{n,N}$ for $n = 2$ and $n = 3$.

**Proposition 5.5.** *For $n = 2, 3$ the coefficients in the decomposition (5.10) are*

$$\begin{aligned} &a_{2,N} = b_{2,N} = 1 \\ &a^2_{3,N} = 1 + 3N^{-2} \ , \ b^2_{3,N} = 2 \, . \end{aligned} \tag{5.26}$$

**Proof for $n = 2$**

We have $\tilde{\alpha} = (0)(12)(34)$, $\pi \in \mathcal{P}_2(2,2)$

| $\pi$ | $\pi\tilde{\alpha}$ | $\mathrm{cyc}_0$ | $\mathcal{P}'$ |
|---|---|---|---|
| (13)(24) | (0)(14)(23) | 2 | 0 |
| (14)(23) | (0)(13)(24) | 2 | 1 |

**Proof for $n = 3$**

We have $\tilde{\alpha} = (0)(123)(456), \pi \in \mathcal{P}_2(3,3)$

| $\pi$ | $\pi\tilde{\alpha}$ | $\mathrm{cyc}_0$ | $\mathcal{P}'$ |
|---|---|---|---|
| (14)(25)(36) | (0)(153426) | 1 | 1 |
| (14)(26)(35) | (0)(16)(25)(34) | 3 | 0 |
| (15)(24)(36) | (0)(14)(26)(35) | 3 | 0 |
| (15)(26)(34) | (0)(163524) | 1 | 1 |
| (16)(24)(35) | (0)(143625) | 1 | 1 |
| (16)(25)(34) | (0)(15)(24)(36) | 3 | 1 |

## 5.5 Asymptotics $N \to \infty$

### 5.5.1 The coefficients $K_{k,n,N}$

Let us recall that

$$K_{k,n,N} = \sum_{\pi \in \Pi_{n,k}} C_\pi(\alpha_{nk}) \text{ where } C_\pi(\alpha_{nk}) = \left(\frac{1}{N}\right)^{nk/2 - \mathrm{cyc}_0(\pi\alpha_{nk})}$$

.

**Proposition 5.6.** *1. We have*

$$\mathrm{cyc}_0(\pi\alpha_{nk}) \le \frac{nk}{2} \tag{5.27}$$

*with equality if and only if the partition $\pi$ in $\Pi_{n,k}$ is non crossing, i.e. the partition on the $n$-blocks of $\pi$ is a non-crossing partition of $[k]$ and satisfies the condition (5.29) below.*

*2.*

$$\lim_N K_{k,n,N} = C_{k/2}. \tag{5.28}$$

*where $C_n$ denotes the nth Catalan number.*

**Proof:** Let $k$ even and $\pi \in \Pi_{n,k}$.
We denote by $\bar{1}, \dots, \bar{k}$ the successive blocks of $[nk]$ ($\bar{j} = \{(j-1)n+1, \dots, jn\}$). A partition $\pi \in \Pi_{n,k}$ induces a pair partition $\bar{\pi}$ on the $k$ blocks $\bar{j}$ in $k/2$ pairs. Let

$$\alpha = \alpha_{nk} = (01 \dots kn)$$

and denote by $C_0$ the cycle of $\pi\alpha_{nk}$ contening 0. Then, $nk$ belongs to $C_0$. Therefore, the number of elements of $[nk]$, not belonging in $C_0$ is less than $nk-1$, and if $\pi\alpha$ has no singletons,we have :

$$\mathrm{cyc}_0(\pi\alpha_{nk}) < nk/2\,.$$

Assume now that $\pi\alpha_{nk}$ has a singleton. It implies that the partition $\bar{\pi}$ has an interval $\{\bar{j},\overline{j+1}\}$ for some $1\le j\le k-1$ and $(jn, jn+1)\in\pi$. The singleton of $\pi\alpha_{nk}$ is $\{jn\}$ and there is no other singletons among the interval $[(j-1)n+1,(j+1)n-1]$. The restriction of $\pi\alpha_{nk}$ on this interval has thus at most $n$ cycles and the maximum of $n$ cycles occurs with a singleton and $n-1$ pairs.
If the restriction of $\pi\alpha_{nk}$ on the above interval has a pair $(x,y)$ with $x<y$, it means that $x\le jn-1$, $y\ge jn+2$ and $\pi(x+1)=y,\ \pi(y+1)=x$. Thus, the only possibility for the restriction of $\pi$ to have $n$ cycles (a singleton and $(n-1)$ pairs) is that $\pi$ on this interval is non crossing, i.e. satisifies :

$$\pi((j-1)n+r)=(j+1)n+1-r,\ 1\le r\le n;$$

We now prove that $(j-1)n$ and $(j+1)n$ are in the same cycle $C_j$ of $\pi\alpha_{nk}$. Note that $a:=(\pi\alpha)^{(-1)}((j-1)n)$ does not belong to $(\bar{j}\cup\overline{j+1})$. The image $\pi\alpha((j-1)n)=\pi((j-1)n+1)\in(\bar{j}\cup\overline{j+1})$. The only way to pass from $\pi\alpha((j-1)n)$ to $a$ in the cycle $C_j$ (i.e. to exit $\bar{j}\cup\overline{j+1}$) is to pass by the point $(j+1)n$. We can thus "identify" $(j-1)n$ and $(j+1)n$ to count the number of cycles. Thus, we are led to the study of the restriction of $\pi$ on $[kn]\backslash(\bar{j}\cup\overline{j+1})$ with $\tilde{\alpha_{nk}}=(012\ldots(j-1)n\ (j+1)n+1\ldots kn)$. A recursive argument on $k$ gives $\mathrm{cyc}_0(\pi\alpha_{nk})\le nk/2$. Moreover, the partitions $\pi$ giving the maximum number of cycles $nk/2$ are those for which $\bar{\pi}$ is non crossing (recall than a NC partition contains at least one interval $(\bar{j},\overline{j+1})$) and satisfies the following condition : if $(\bar{j},\bar{l})$, $j<l$ is a pair of $\bar{\pi}$, then,

$$\pi((j-1)n+r)=ln+1-r,\ 1\le r\le n. \tag{5.29}$$

For such a partition, $\lim_N C_\pi(\alpha_{nk})=1$ and $\lim_N C_\pi(\alpha_{nk})=0$ in the other cases. The number of partitions $\pi$ in $\Pi_{n,k}$ giving the maximal number of cycles is thus equal to the number of non crossing pair partitions $\bar{\pi}$, that is the Catalan number $C_{k/2}$, proving (5.28). □

### 5.5.2 The coefficents $a_{n,N}$ and $b_{n,N}$

To echo the asymptotics of the moments of $\mu_{M_\infty}$ given in (5.9)-(5.28), we have the following statement on the asymptotics of coefficients $a_{n,N}$ and $b_{n,N}$ of the decomposition (5.10) and its consequence on $\mu_{M_\infty}$.

**Proposition 5.7.** *1. The coefficients $a_{n,N}$ and $b_{n,N}$ satisfy :*

$$a_{n,N}^2 = 1 + O(N^{-1}) \ , \quad b_{n,N}^2 = n - 1 + O(N^{-1})\,. \tag{5.30}$$

*2. As $N \to \infty$, the empirical spectral distribution of $M_\infty$ converges in distribution to* $\mathrm{SC}(0; \sigma_n^2)$.

The first statement will be a consequence of the following Lemma 5.8. For the second one, it is enough to say that for fixed $n$,

- the empirical spectral distribution of $\sigma_n b_{n,N} \xi I_N$ converges to $\delta_0$,
- the empirical spectral distribution of $\sigma_n a_{n,N}\,\mathrm{GUE}(N^{-1})$ converges to $\mathrm{SC}(0; \sigma_n^2)$
- $\sigma_n b_{n,N} \xi I_N$ and $\sigma_n a_{n,N}\,\mathrm{GUE}(N^{-1})$ are independent.

**Proof of Proposition 5.7 1**

Going back to the end of Sec. 5.3, we see that if $\pi \in \mathcal{P}_2(n,n)$, then $\pi\tilde{\alpha}$ has no singleton. Therefore

$$\mathrm{Tr}_{\pi\tilde{\alpha}}(I_N, \dots, I_N, A, I_N, \dots, A) = O(N^n)$$

and the term of order $N^n$ is obtained when $\pi\tilde{\alpha}$ is a product of $n$ pairs ($\times(0)$). We study those pairings $\pi$ such that $\pi\tilde{\alpha} = (0)\prod_{i=1}^n \tau_i$ where $\tau_i$ is a pair (or a transposition). Two cases appear :
1) one of the $\tau$ is $(n, 2n)$. In this case,

$$\mathrm{Tr}_{\pi\tilde{\alpha}}(I_N, \dots, I_N, A, I_N, \dots, A) = N^{n-1}\mathrm{Tr}\,(A^2) = N^n \mathrm{tr}(A^2).$$

2) $n$ and $2n$ are not in the same pair of $\pi$. Then,

$$\mathrm{Tr}_{\pi\tilde{\alpha}}(I_N, \dots, I_N, A, I_N, \dots, A) = N^{n-2}(\mathrm{Tr}\,(A))^2 = N^n (\mathrm{tr}(A))^2.$$

**Proposition 5.8.** *1) There is only one partition $\pi$, the non crossing partition $\pi = (1, 2n)(2, 2n-1)\dots(n, n+1)$ of $\mathcal{P}_2(n,n)$, such that*

$$\mathrm{Tr}_{\pi\tilde{\alpha}}(I_N, \dots, I_N, A, I_N, \dots, A) = N^n \mathrm{tr}(A^2).$$

*2) There are $n-1$ pair partitions $\pi \in \mathcal{P}_2(n,n)$ such that :*

$$\mathrm{Tr}_{\pi\tilde{\alpha}}(I_N, \dots, I_N, A, I_N, \dots, A) = N^n (\mathrm{tr}(A))^2.$$

**Proof of Proposition 5.8**

1. We assume that $\pi\tilde{\alpha} = (1, j_1)(2, j_2)\dots(n-1, j_{n-1})(n, 2n)$. Note that

$$\tilde{\alpha}^{-1} = (0)(n\ (n-1)\dots 1)(2n\ (2n-1)\dots n+1).$$

We have sucessively

$$\begin{aligned} \pi(1) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(1) = \pi\tilde{\alpha}(n) = 2n \\ \pi(2n) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(2n) = \pi\tilde{\alpha}(2n-1)\,, \end{aligned} \tag{5.31}$$

but since $\pi$ is a pairing $\pi(2n) = 1$ so that

$$\pi\tilde{\alpha}(2n-1) = 1\,, \tag{5.32}$$

which implies $j_1 = 2n-1$. Then, similarly

$$\begin{aligned} \pi(2) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(2) = \pi\tilde{\alpha}(1) = j_1 = 2n-1 \\ 2 = \pi(2n-1) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(2n-1) = \pi\tilde{\alpha}(2n-2) \\ j_2 &= 2n-2 \end{aligned} \tag{5.33}$$

and so on. Eventually

$$\pi = (1, 2n)(2, 2n-1)\dots(n, n+1) \tag{5.34}$$

so that there is one and only one $\pi$ giving the dominant term of $\mathrm{tr}(A^2)$.

2. We now assume that

$$\pi\tilde{\alpha} = (i_1, j_1)\dots(i_{n-2}, j_{n-2})(n, j_{n-1})(i_n, 2n)$$

We set

$$\pi(n) = \pi\tilde{\alpha}(n-1) =: k \in [n+1, 2n]$$

If $k = n+1$ we have

$$\begin{aligned} \pi(n) = n+1 &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n) = \pi\tilde{\alpha}(n-1) \\ \pi(n+1) = n &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n+1) = \pi\tilde{\alpha}(2n) \end{aligned}$$

This is a contradiction since, by assumption, $(n, 2n)$ is not a pair of $\pi\tilde{\alpha}$.

Therefore, $k = n+1+j$ for $j \in [1, n-1]$ and we have successively:

$$\begin{aligned} \pi(n) = n+1+j &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n) = \pi\tilde{\alpha}(n-1) \\ \pi(n+j+1) = n &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n+j+1) = \pi\tilde{\alpha}(n+j) \end{aligned}$$

Thus $(n, n+j) \in \pi\tilde{\alpha}$ and

$$\pi(1) = \pi\tilde{\alpha}\tilde{\alpha}^{-1}(1) = \pi\tilde{\alpha}(n) = n + j.$$

implying that $1 = \pi(n+j) = \pi\tilde{\alpha}(n+j-1)$.
Thus $\pi(2) = \pi\tilde{\alpha}(1) = n+j-1$. We obtain successively:

$$\begin{aligned}
\pi(l) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(l-1) = n+j-l+1,\ l \le j \\
&\cdots \\
\pi(j) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(j-1) = n+1 \\
\pi(n+1) &= j = \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n+1) = \pi\tilde{\alpha}(2n) \\
\pi(j+1) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(j+1) = \pi\tilde{\alpha}(j) = 2n \\
\pi(2n) &= j+1 = \pi\tilde{\alpha}\tilde{\alpha}^{-1}(2n) = \pi\tilde{\alpha}(2n-1) \\
&\cdots \\
\pi(n-1) &= \pi\tilde{\alpha}\tilde{\alpha}^{-1}(n-2) = n+j+2\,.
\end{aligned}$$

and eventually

$$\pi = (1, n{+}j)(2, n{+}j{-}1)\dots(j, n{+}1)(j{+}1, 2n)(j{+}2, 2n{-}1)\dots(n, n{+}j{+}1)\,.$$

In conclusion, there are exactly $n-1$ pairings giving the dominant term in $\mathrm{tr}(A)^2$.

# 6 Matrix-variate Hermite polynomials

## 6.1 Introduction

In this section, we consider two models of matrix valued processes $X(t)$ obtained by smoothing a matrix Brownian motion. Like in the previous sections, we will study the asymptotic behavior of $\varepsilon^{-1/2}\int_0^{1/\varepsilon} F(X(t)dt$, but now $F$ will be a convenient real function such that $F(X(t))$ is centered. The classical key is the decomposition according to some Hermite-like basis, called the matrix-variate Hermite polynomials.

In the first model, we consider the bilateral Brownian motion $\mathcal{W}^{(N)}(t)$ on the space $S_N$ of $N \times N$ symmetric real matrices (a.k.a. GOE(1)) and a smoothing real function $\varphi$ such that $\|\varphi\|_2 = 1$.

In the second model, we consider the bilateral Brownian motion $\mathcal{W}^{\ell,N}(t)$ on the space $\mathbb{R}^{\ell\times N}$ of $\ell \times N$ rectangular real matrices and a smoothing function $\Phi$ living in the space of $N \times N$ matrices and satisfying

$$\int \Phi(s)^T \Phi(s) ds = \mathrm{Id}_N\,. \tag{6.1}$$

The smoothed processes are defined respectively by

$$X(t) = \int_{-\infty}^{\infty} \varphi(t-s)d\mathcal{W}^{(N)}(s) \in S_N \,, \tag{6.2}$$

$$X(t) = \int_{-\infty}^{\infty} d\mathcal{W}^{(\ell,N)}(s)\Phi(t-s) \in \mathbb{R}^{\ell\times N} \,. \tag{6.3}$$

These processses are stationary Gaussian and the covariances are given respectively by

$$\mathbb{E}\left(X(t_2)_{ij}X(t_1)_{rs}\right) = \rho(t_2-t_1)\left(\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}\right)\,, \tag{6.4}$$

where $\rho$ was defined in (2.5) and

$$\mathbb{E}\left(X(t_2)_{ij}X(t_1)_{rs}\right) = \delta_{ir}R(t_2-t_1)_{js} \tag{6.5}$$

where

$$R(t) = \int \Phi(s)^T\Phi(t+s)ds \in \mathbb{R}^{N\times N} \,. \tag{6.6}$$

The definitions of matrix-variate Hermite polynomials in the symmetric and rectangular cases are given in the following subsections. They need a recall on zonal polynomials, as in [12].

Let $Z$ be a symmetric $m\times m$ matrix, $k$ be an integer and $\kappa$ be a partition of $k$ in no more than $m$ parts denoted $\kappa \vdash k$, i.e. $\kappa = (k_1,\ldots,k_l)$, $l \leq m$, $k_1 \geq k_2 \geq k_l \geq 1$, and $k = k_1+\cdots+k_l$. The zonal polynomials $C_\kappa(Z)$ are defined as a basis of the space of all homogeneous symmetric polynomials in the latent roots of $Z$.

Actually, to simplify the exposition, the main results, in Propositions 6.1 and 6.4 are stated for polynomials only.

## 6.2 Symmetric real matrices

The Hermite polynomials $\mathbf{H}_\kappa^{(N)}(X), \kappa \vdash k = 0,1,\cdots$, where $\kappa$ is a partition in no more than $N$ parts, constitute a complete system of orthogonal polynomials associated with the distribution of density

$$\phi_N(X) = 2^{-N/2}\pi^{-N(N+1)/4}\exp -\frac{1}{2}\mathrm{Tr}\,(X^2) \tag{6.7}$$

with respect to the Lebesgue measure on $\mathbb{R}^{N(N+1)/2}$. Actually

$$\int_{S_N} \mathbf{H}_\kappa^{(N)}(X)\mathbf{H}_\sigma^{(N)}(X)\phi_N(X)\mathrm{d}X = \delta_{\kappa,\sigma}k!C_\kappa(\mathrm{Id}_N) \tag{6.8}$$

([18, formula (2.13)], [13, formula 5.3]).

We assume that $\mathcal{W}^{(N)}(t)$ is a real symmetric matrix-valued Brownian process, so that $X(t)$ defined in (6.2) is a stationary process with 1d-marginal density $\phi_N$. We set

$$\zeta_\varepsilon^{(\kappa)} = \varepsilon^{1/2} \int_0^{\varepsilon^{-1}} \mathbf{H}_\kappa^{(N)}(X(t))dt\,. \tag{6.9}$$

The main result of this section is the following proposition.

**Proposition 6.1.** *As $\varepsilon \to 0$,*

$$\zeta_\varepsilon^{(\kappa)} \xrightarrow[\varepsilon\to 0]{(d)} \mathcal{N}(0; \sigma^2_{N,\kappa}) \tag{6.10}$$

*where*

$$\sigma^2_{N,\kappa} = k! C_\kappa(\mathrm{Id}_N) \int_0^\infty \rho(t)^k dt\,,$$

*and $\rho$ was defined in a previous section.*

*Proof.* We skip the upper index $^{(N)}$. The centered process $(\mathbf{H}_\kappa(X(t)), t \geq 0)$ is a non-linear function of a Gaussian multi-dimensional process and we apply the Breuer-Major theorem. We will prove that the covariance of the process $\mathbf{H}_\kappa(t)$ defined by

$$\Gamma(t) = \mathbb{E}\left(\mathbf{H}_\kappa(X(t))\mathbf{H}_\kappa(X(0))\right)$$

satisfies

$$\Gamma(t) = \rho(t)^k k! C_\kappa(\mathrm{Id}_N)\,. \tag{6.11}$$

If $t$ is fixed, we observe the identity in distribution

$$(X(t), X(0)) \overset{(d)}{=} \left(\rho(t)X(0) + \sqrt{1-\rho(t)^2}X', X(0)\right) \tag{6.12}$$

where $X(0)$ and $X'$ are independent and have density $\phi_N$.

Summarizing, we have to compute

$$\mathbb{E}\left(\mathbf{H}_\kappa(X)\mathbf{H}_\kappa(Y)\right)$$

with

$$Y = \rho X + \sqrt{1-\rho^2}X'\ ,\ X \perp X', X \overset{(d)}{=} X'\,. \tag{6.13}$$

We will condition upon $X$ and use a reduction to a computation coordinate-wise. We will use the following lemma, which is probably well known.

**Lemma 6.2.** *There exists a system of numerical constants* $u^\kappa_\alpha$ *indexed by* $\kappa$*, a partition of* $k$ *and* $\alpha = (\alpha_{ij} \in \mathbb{N}^{\frac{N(N+1]}{2}})$*, a multi-index with* $|\alpha| = \sum_{i\leq j} \alpha_{ij} = k$ *such that*

$$\mathbf{H}_\kappa(X) = \sum_{|\alpha|=k} u^\kappa_\alpha \left( \prod_i 2^{-\alpha_{ii}/2} H_{\alpha_{ii}}(X_{ii}\sqrt{2}) \right) \prod_{i<j} H_{\alpha_{ij}}(X_{ij})\,, \tag{6.14}$$

*where the polynomials* $H_p$*,* $p \geq 1$ *are the classical (scalar) Hermite polynomials.*

*Proof.* The proof is based on the Rodrigues formula in the symmetric case is [12, Sec. 3.2]. It says that if

$$\partial X =: \left( \frac{1}{2}(1+\delta_{ij}) \frac{\partial}{\partial x_{ij}} \right)_{i,j} .$$

then

$$\mathbf{H}_\kappa(X) = \phi_N(X)^{-1} C_\kappa(-\partial X) \phi_N(X)\,. \tag{6.15}$$

Besides we know that, if $Z \in S_N$ then by definition, $C_\kappa(Z)$ is a symmetric homogeneous polynomial in the eigenvalues. It has then a decomposition

$$C_\kappa(Z) = \sum_\nu c_{\kappa\nu}\ s_1(Z)^{\nu_1} \ldots s_k(Z)^{\nu_k}\,,$$

where $s_j(Z) = \mathrm{tr} Z^j$ and $\nu = (1^{\nu_1} 2^{\nu_2} \ldots k^{\nu_k})$ with $\sum_j j\nu_j = k$. Since $t_j(Z)^{\nu_j}$ is a homogeneous polynomial of degree $j\nu_j$ in the entries of $Z$, we conclude that $C_\kappa(Z)$ has a decomposition

$$C_\kappa(Z) = \sum_{|\alpha|=k} u^\kappa_\alpha \prod_{i\leq j} Z_{ij}^{\alpha_{ij}}\,, \tag{6.16}$$

(see also [19, formula 2.6]).

This leads to

$$\mathbf{H}_\kappa(X) = \phi_N(X)^{-1} \sum_{|\alpha|=k} (-1)^k u^\kappa_\alpha \prod_{i<j} 2^{-\alpha_{ij}} \frac{\partial^{\alpha_{ij}}}{\partial X_{ij}^{\alpha_{ij}}} \quad \prod_i \frac{\partial^{\alpha_{ii}}}{\partial X_{ii}^{\alpha_{ii}}} \phi_N(X)\,.$$

Now, the obvious decomposition

$$\phi_N(X) = \prod_i \phi_1(X_{ii}) \prod_{i<j} \phi_1(X_{ij}\sqrt{2})\,,$$

gives

$$\mathbf{H}_\kappa(X) = \sum_{|\alpha|=k} (-1)^k u_\alpha^\kappa \prod_{i<j} 2^{-\alpha_{ij}} \phi_1(X_{ij}\sqrt{2})^{-1} \frac{\partial^{\alpha_{ij}}}{\partial X_{ij}^{\alpha_{ij}}} \phi_1(X_{ij}\sqrt{2})$$
$$\times \prod_i \phi_1(X_{ii})^{-1} \frac{\partial^{\alpha_{ii}}}{\partial X_{ii}^{\alpha_{ii}}} \phi_1(X_{ii}) \tag{6.17}$$

Then, using the classical definition of classical Hermite polynomials :

$$\phi_1(x)^{-1} \frac{d^p}{dx^p} \phi_1(x) = (-1)^p H_p(x) \,,$$

we get (6.14).

End of the proof of Prop. 6.1

In the setting of (6.13) and using the property of the Ornstein-Uhlenbeck semigroup, we get

$$\mathbb{E}\left[H_{\alpha_{ij}}(\rho X_{ij} + \sqrt{1-\rho^2} X'_{ij})|X\right] = \rho^{\alpha_{ij}} H_{\alpha_{ij}}(X_{ij}) \,. \tag{6.18}$$

A plugging into (6.14) leads to

$$\begin{aligned} \mathbb{E}[\mathbf{H}_\kappa(Y)|X] &= \sum_{|\alpha|=k} u_\alpha^\kappa \left(\prod_i 2^{-\alpha_{ii}/2} \rho^{\alpha_{ii}} H_{\alpha_{ii}}(X_{ii})\right) \prod_{i<j} \rho^{\alpha_{ij}} H_{\alpha_{ij}}(X_{ij}) \\ &= \rho^k \sum_{|\alpha|=k} u_\alpha^\kappa \left(\prod_i 2^{-\alpha_{ii}/2} H_{\alpha_{ii}}(X_{ii})\right) \prod_{i<j} H_{\alpha_{ij}}(X_{ij}) \\ &= \rho^k \mathbf{H}_\kappa(X) \,, \end{aligned} \tag{6.19}$$

and this proves that (6.43) holds true.

*Remark* 4. Another proof of the Lemma could be based on a striking representation [13, 5.48 p. 87] :

$$\mathbf{H}_\kappa(X) = \int_{S_N} C_\kappa(X + iM)\phi_N(M)\mathrm{d}M \,. \tag{6.20}$$

The second ingredient would be the decomposition (6.16) and independence of coordinates. The issue is that we don't know any definition of $C_\kappa$ applied to a complex (although symmetric) matrix.

### 6.3 Hermitian matrices

A similar analysis could be made for Hermitian matrices as well. We do not give details, but present some examples.

The following proposition gives an explicit way to compute the matrix-variate Hermite polynomials in the Hermitian context. Following the notations therein, if $\kappa$ is a partition of $n$, let us denote by $\chi^\kappa$ the character of the irreducible representation of $S(n)$ corresponding to $\kappa$ and let us identify $\chi^\kappa$ with the element

$$\sum_{\sigma\in S(n)} \chi^\kappa(\sigma)\sigma \in \mathbb{C}[S(n)]\,.$$

**Proposition 6.3.** *[3, Cor. 6.11] The Hermite polynomial of matrix argument $X$, indexed by a partition $\kappa$, is a multiple of*

$$\widetilde{\mathbf{H}}_\kappa^{(N)}(X) := \sum_{\sigma\in S(n)} \chi^\kappa(\sigma)\tilde{H}_{\hat\sigma}(X,N) \tag{6.21}$$

*where $\tilde{H}_\alpha(X,u)$ is defined by* (5.4)[2] *and $\hat\sigma \in S_0(n)$ is the permutation $\sigma$ with the additional cycle* (0).

To begin with, let us consider the case $n = 2$.

1) $\hat\sigma = (0)(12)$

| $\pi$ | $2-\lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 2 | (0)(12) | (0)(12) | $\mathrm{Tr}\,X^2$ | 0 | 1 |
| (12) | 1 | (0)(1)(2) | (0) | 1 | 1 | $N$ |

hence

$$\tilde{H}_{\hat\sigma}(X,N) = \mathrm{Tr}\,(X^2) - N^2\,. \tag{6.22}$$

2) $\hat\sigma = \mathrm{id}$

| $\pi$ | $2-\lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 2 | id | id | $(\mathrm{Tr}\,X)^2$ | 0 | 1 |
| (12) | 1 | (0)(12) | (0) | 1 | 1 | 1 |

hence

$$\tilde{H}_{\hat\sigma}(X,N) = (\mathrm{Tr}\,X)^2 - N\,. \tag{6.23}$$

The coefficients $\chi^\kappa(\sigma)$ associated to the partitions (or Young tableaux) are given in [15].

If $\kappa = (1,1)$ then $\chi^\kappa = \mathrm{id} - (12)$ so that

$$\widetilde{\mathbf{H}}_\kappa(X) = \left((\mathrm{Tr}\,X)^2 - N\right) - \left(\mathrm{Tr}\,(X^2) - N^2\right)\,. \tag{6.24}$$

[2]In [3, Cor. 6.11], the above condition $u = N$ was improperly replaced by $u = 1/N$.

If $\kappa = (2)$ then $\chi^\kappa = \mathrm{id} + (12)$ so that

$$\widetilde{\mathbf{H}}_\kappa(X) = \big((\mathrm{Tr}\, X)^2 - N\big) + \big(\mathrm{Tr}\,(X^2) - N^2\big)\ . \tag{6.25}$$

We can check easily that these polynomials are orthogonal under $\phi_N$.

Let us look at the case $n = 3$.

1) $\hat\sigma = (0)(123)$

| $\pi$ | $3 - \lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 3 | (0)(123) | (0)(123) | $\mathrm{Tr}\, X^3$ | 0 | 1 |
| (12)(3) | 2 | (0)(1)(23) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |
| (13)(2) | 2 | (0)(12)(3) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |
| (1)(23) | 2 | (0)(13)(2) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |

hence

$$\tilde H_{\hat\sigma}(X, N) = -\mathrm{Tr}\, X^3 + 3N \mathrm{Tr}\, X\,. \tag{6.26}$$

2) $\hat\sigma = (0)(12)(3)$

| $\pi$ | $3 - \lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 3 | (0)(12)(3) | (0)(12)(3) | $\mathrm{Tr}\, X^2 \mathrm{Tr}\, X$ | 0 | 1 |
| (12)(3) | 2 | id | (0)(1) | $\mathrm{Tr}\, X$ | 1 | $N$ |
| (1)(23) | 2 | (0)(132) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | $N^{-1}$ |
| (13)(2) | 2 | (0)(123) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | $N^{-1}$ |

hence

$$\tilde H_{\hat\sigma}(X, N) = (N^2 + 2)\mathrm{Tr}\, X - (\mathrm{Tr}\, X^2)\mathrm{Tr}\, X\,. \tag{6.27}$$

3) $\hat\sigma = \mathrm{id}$

| $\pi$ | $3 - \lvert\pi\rvert$ | $\pi\alpha$ | $\beta$ | $\mathrm{Tr}_\beta$ | $l$ | $q^c$ |
|---|---|---|---|---|---|---|
| id | 3 | id | id | $(\mathrm{Tr}\, X)^3$ | 0 | 1 |
| (12)(3) | 2 | (0)(12)(3) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |
| (13)(2) | 2 | (0)(13)(2) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |
| (1)(23)) | 2 | (0)(1)(23) | (0)(1) | $\mathrm{Tr}\, X$ | 1 | 1 |

hence

$$\tilde H_{\hat\sigma}(X, N) = 3N\mathrm{Tr}\, X - (\mathrm{Tr}\, X)^3\,. \tag{6.28}$$

The coefficients $\chi^\kappa(\sigma)$ associated to the partitions (or Young tableaux) are given in [15, Ex. 6 p. 14 and Example 4.5 p.47].

1. If $\kappa = (1,1,1)$ then $\chi^\kappa = \mathrm{id} - (12) - (13) - (23) + (123) + (132)$ so that owing to (6.26-6.27-6.28)

$$\widetilde{\mathbf{H}}^{(N)}_\kappa(X) = -(\mathrm{Tr}\, X^3) + 3(\mathrm{Tr}\, X)(\mathrm{Tr}\, X^2) - 2\mathrm{Tr}\, X^3 - 3(N-1)(N-2)\mathrm{Tr}\, X\,. \tag{6.29}$$

2. If $\kappa = (2,1)]$, then $\chi^\lambda = 2\mathrm{id} - (123) - (132)$ so that owing to (6.26-6.28)

$$\widetilde{\mathbf{H}}_\kappa^{(N)}(X) = 2\left(\mathrm{Tr}\,(X^3) - (\mathrm{Tr}\,X)^3\right) . \tag{6.30}$$

3. If $\kappa = (3)$ then $\chi_\lambda = \mathrm{id} + (12) + (13) + (23) + (123) + (132)$ so that owing to (6.26-6.27-6.28)

$$\widetilde{\mathbf{H}}_\kappa^{(N)}(X) = -(\mathrm{Tr}\,X)^3 - 3\mathrm{Tr}\,X\mathrm{Tr}\,X^2 - 2\mathrm{Tr}\,X^3 + 3(N+1)(N+2)\mathrm{Tr}\,X . \tag{6.31}$$

These results are consistent with the expressions of Hermite polynomials given in [14]. Therein, the complex case corresponds to $\alpha = 2/\beta = 1$, $C^1_\kappa$ are the complex zonal polynomials and $s_i := \mathrm{Tr}\,(X^i)$.

$$\begin{aligned} H^1_{(1,1)} &= C^1_{(1,1)} + \frac{N(N-1)}{2} \\ &= \frac{1}{2}\left(s_1^2 - s_2 + N(N-1)\right) . && (6.32) \\ H^1_{(2)} &= C^1_{(2)} - N(N+1) \\ &= s_1^2 + s_2 - N(N+1) . && (6.33) \\ H^1_{(1,1,1)} &= C^1_{(1,1,1)} + \frac{(N-1)(N-2)}{2}C^1_{(1)} \\ &= \frac{1}{6}s_1^3 - \frac{1}{2}s_1 s_2 + \frac{1}{3}s_3 + \frac{(N-1)(N-2)}{2}s_1 . && (6.34) \\ H^1_{(2,1)} &= C^1_{(2,1)} + 0\ C^1_{(1)} \\ &= \frac{1}{3}\left(s_1^3 - s_3\right) . && (6.35) \\ H^1_{(3)} &= C^1_{(3)} + \frac{(N+1)(N+2)}{2}C^1_{(1)} \\ &= \frac{1}{6}s_1^3 + \frac{1}{2}s_1 s_2 + \frac{1}{3}s_3 + \frac{(N+1)(N+2)}{2}s_1 . && (6.36) \end{aligned}$$

## 6.4 Rectangular matrices

We assume in this part that $\ell \le N$. The Hermite polynomials $\mathbf{H}_\kappa^{(\ell,N)}, \kappa \vdash k = 0,1,\dots,$ where $\kappa$ is a partition in no more than $\ell$ parts, and whose variates are rectangular $\ell \times N$ real matrices constitue a complete system of orthogonal polynomials associated with the distribution $\mathcal{N}_{\ell\times N}(0, \mathrm{Id}_\ell \otimes \mathrm{Id}_N)$ of density

$$\phi_{\ell,N}(X) = (2\pi)^{-\ell N/2} \exp -\frac{1}{2}\mathrm{Tr}\,(XX^T) \tag{6.37}$$

with respect to the Lebesgue measure on $\mathbb{R}^{\ell N}$. Actually

$$\int_{\mathbb{R}^{\ell N}} \mathbf{H}_\kappa^{(\ell,N)}(X)\mathbf{H}_\sigma^{(\ell,N)}(X)\phi_{\ell,N}(X) = \delta_{\kappa,\sigma}4^k \left(\frac{N}{2}\right)_\kappa k! C_\kappa(\mathrm{Id}_\ell)\,. \tag{6.38}$$

We refer to [12, Sec. 4] and [19] for the notations and for the interest of this model.

We assume that $\mathcal{W}^{(\ell,N)}(t)$ is a $\ell \times N$ matrix-valued process whose coordinates are independent bilateral standard Brownian motions. Notice that owing to the assumptions (6.1) and (6.5), the stationary process $X(t)$ defined in (6.3) has marginals distributed as $\mathcal{N}_{\ell\times N}(0, \mathrm{Id}_\ell \otimes \mathrm{Id}_N)$.

Recall the definition (6.1)

$$R(t) = \int \Phi(s)^T \Phi(t+s) ds$$

which satisfies by assumption (6.1) $R(0) = \mathrm{I}d_N$. In this context the correlation function $\rho$ will be replaced by the symmetric, semidefinite positive matrix

$$\mathbf{R}(t) = (R(t)R(t)^T)^{1/2}\,. \tag{6.39}$$

Actually, we have like (2.6)

$$\mathbf{R}(t) \le \mathrm{I}d_N\,. \tag{6.40}$$

It is a consequence of the matrix Cauchy-Schwarz inequality - see for instance [1, formula (2.9)]- which says that if $|A| := (A^T A)^{1/2}$ then

$$\left|\sum_1^m B_i A_i\right| \le \left(\sum_1^m |A_i|^2\right)$$

whenever $\sum_1^m |B_i^T|^2 \le \mathbf{1}$. This inequality can be easily extended to integrals instead of sums, and taking $A_i \equiv \Phi(s)^T$, $B_i \equiv \Phi(t+s)$ we get exactly $|R(t)| \le \mathrm{Id}_N$ hence (6.40) since $R(t)R(t)^T$ and $R(t)^T R(t)$ have the same nonzero eigenvalues.

We set

$$\eta_\varepsilon^{(\kappa)} = \varepsilon^{-1/2} \int_0^{1/\varepsilon} \mathbf{H}_\kappa^{(\ell,N)}(X(t))dt\,, \tag{6.41}$$

The main result of this section is the following proposition.

**Proposition 6.4.** *As* $\varepsilon \to 0$

$$\eta_\varepsilon^\kappa \xrightarrow[\varepsilon\to 0]{(d)} \mathcal{N}(0;\sigma^2_{\ell,N,\kappa}) \tag{6.42}$$

*where*

$$\sigma^2_{\ell,N,\kappa} = 4^{-k}k!\left(\frac{N}{2}\right)_\kappa \frac{C_\kappa(\mathrm{Id}_\ell)}{C_\kappa(\mathrm{Id}_N)}\int_0^\infty C_\kappa(\mathbf{R}^2(t))\mathrm{d}t\,.$$

*Proof.* In the sequel, we skip the index $^{(\ell,N)}$ for simplicity.

Since $X(t)$ is a standard Gaussian matrix and since $\mathbf{H}_0 \equiv 1$, the process $(\mathbf{H}_\kappa(X(t)), t \geq 0)$ is centered. Actually it is is a non-linear function of a Gaussian multi-dimensional real process. We apply a continuous version of the Breuer-Major theorem. We have to prove that the covariance of $\mathbf{H}_\kappa(t)$ satisfies

$$\mathbb{E}\left(\mathbf{H}_\kappa(X(t))\mathbf{H}_\kappa(X(0)\right) = 4^{-k}k!\left(\frac{N}{2}\right)_\kappa \frac{C_\kappa(\mathrm{Id}_\ell)}{C_\kappa(\mathrm{Id}_N)}C_\kappa(\mathbf{R}^2(t))\,. \tag{6.43}$$

We continue as in the end of proof of Prop. 6.1, with the help of [19, Th. 3.12], but with a careful treatment of $R(t)$ which is not necessarily symmetric.

**Lemma 6.5.** *Set* $\mathbf{S}(t) = \left(\mathrm{Id}_N - \mathbf{R}(t)^2\right)^{1/2}$. *Then for fixed* $t$,

$$(X(t), X(0)) \overset{(d)}{=} (Y, X(0))\ , \tag{6.44}$$

*where* $Y := X(0)R(t)^T + X'\mathbf{S}(t)$ *with* $X' \overset{(d)}{=} X(0)$ *and independent of* $X(0)$.

*Proof.* We have, by independence

$$\begin{aligned}
\mathbb{E}(Y_{ij}Y_{rs}) &= \mathbb{E}\left((X(0)R(t)^T)_{ij}(X(0)R(t)^T)_{rs}\right) + \mathbb{E}\left((X'S(t))_{ij}(X'S(t))_{rs}\right)\\
&= \sum_{k,p}\left(R(t)_{jk}R(t)_{sp} + \mathbf{S}(t)_{jk}\mathbf{S}(t)_{sp}\right)\delta_{ir}\delta_{kp}\\
&= \left(R(t)R(t)^T + \mathbf{S}(t)^2\right)_{js}\delta_{ir} = \left(\mathbf{R}(t)^2 + \mathbf{S}(t)^2\right)_{ir}\delta_{js} = \delta_{ir}\delta_{js}\,,
\end{aligned}$$

which are the covariances of $X(t)$.

Moreover, again by independence

$$\begin{aligned}
\mathbb{E}\left(Y_{ij}X(0)_{rs}\right) &= \mathbb{E}\left((X(0)R(t)^T)_{ij}X(0)_{rs}\right)\\
&= \sum_k R(t)_{jk}\delta_{ir}\delta_{ks} = R(t)_{js}\delta_{ir}\,,
\end{aligned}$$

which are the cross covariances of $X(t)$ and $X(0)$, as given in (6.5).

End of the proof of Prop. 6.4

From the above lemma, we have

$$\mathbb{E}\left(\mathbf{H}_\kappa(X(t))\mathbf{H}_\kappa(X(0)\right) = \mathbb{E}\left(\mathbf{H}_\kappa(Y)\mathbf{H}_\kappa(X(0)\right)$$

but, since for any $N \times N$ deterministic orthogonal matrix $X(0) \overset{(d)}{=} X(0)U$, we get

$$\mathbb{E}\left(\mathbf{H}_\kappa(X(t))\mathbf{H}_\kappa(X(0)\right) = \mathbb{E}\left(\mathbf{H}_\kappa(X(0)UR(t)^T + X'\mathbf{S}(t))\mathbf{H}_\kappa(X(0)U\right) . \tag{6.45}$$

Now, it is known that $\mathbf{H}_\kappa(MU) = \mathbf{H}_\kappa(U)$ for any $M \in \mathbb{R}^{\ell \times N}$, so that choosing $U = (\mathbf{R}(t))^{-1} R(t)$ (polar decomposition of $R(t)$), we conclude that

$$\mathbb{E}\left(\mathbf{H}_\kappa(X(t))\mathbf{H}_\kappa(X(0)\right) = \mathbb{E}\left(\mathbf{H}_\kappa(X(0)\mathbf{R}(t)^T + X'\mathbf{S}(t))\mathbf{H}_\kappa(X(0)\right) .$$

It is then enough to apply [19, Th. 3.12].

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