Discretization methods for homogeneous fragmentations

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Summary. Homogeneous fragmentations describe the evolution of a unit mass that breaks down randomly into pieces as time passes. They can be thought of as continuous time analogs of a certain type of branching random walks, which suggests the use of time-discretization to shift known results from the theory of branching random walks to the fragmentation setting. In particular, this yields interesting information about the asymptotic behaviour of fragmentations.

On the other hand, homogeneous fragmentations can also be investigated using a powerful technique of discretization of space due to Kingman, namely, the theory of exchangeable partitions of $\mathbb{N}$. Spatial discretization is especially well-suited to develop directly for continuous times the conceptual method of probability tilting of Lyons, Pemantle and Peres.

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1 Introduction

Homogeneous fragmentations form a family of random processes in continuous times which have been introduced in [4]. Roughly, these are particle systems that model a mass that breaks down randomly into pieces as time passes. More precisely, each particle is identified with its mass (i.e. it is specified by a positive real number), and the fragmentation property requires that different particles have independent evolutions. The homogeneity property means that the process started from a single particle with mass $x > 0$ has the same distribution as x times the process started from a single particle with unit mass.

This verbal description has obvious similarities with that of branching random walks. More precisely, let us write $Z^{(t)}$ for the random point measure which assigns a Dirac point mass at $\log x$ for every x varying over the set of particles at time t. Taking logarithms transforms the fragmentation and homogeneity properties into the branching property for random point measures. More precisely, for every $t, t' \geq 0$, $Z^{(t+t')}$ is obtained from $Z^{(t)}$ by replacing each atom $z = \log x$ of $Z^{(t)}$ by a family $\{z + y, y \in \mathcal{Y}\}$, where $\mathcal{Y}$ is distributed as the family of the atoms of $Z^{(t')}$ for $Z^{(0)} = \delta_0$, and distinct atoms z of $Z^{(t)}$ correspond to independent copies of $\mathcal{Y}$.

Homogeneous fragmentations may be seen as extensions of so-called branching random walks in continuous time. The latter have been considered by Uchiyama [27], Biggins [9], Kyprianou [17], ... Their main feature is that each particle has an exponentially distributed lifetime and at the instant of its death, scatters a random number of children-particles in space relative to its death point according to the point process. However the theory of branching processes in continuous time does not encompass homogeneous fragmentations, because usually each fragment starts to split instantaneously, which would correspond to particles with zero lifetime in the branching setting.

The close connection between homogeneous fragmentations and branching random walks suggests that one should try to reduce the study of fragmentations to that of branching random walks by time-discretization. This is the path that we will follow in the first part of this work. Aside from some difficult technical problems (for instance, a most useful notion such as the first branching time has no analog for fragmentations since, in general, dislocations occur instantaneously), this enables us to shift several deep results on branching random walks to the fragmentation setting. In particular, this yields interesting information about the asymptotic behaviour of fragmentations, which refine earlier results in [6].

There is another discretization method that will play an important role in this paper. The fundamental idea is due to Kingman [15], who pointed out that partitions of an object, say with a unit mass, can be fruitfully encoded by partitions of $\mathbb{N}$. In order to explain the coding, we introduce a sequence of i.i.d. random points $U_1, \ldots$ which are picked according to the mass distribution of the object. One then considers at each time $t \geq 0$ the random partition of the set of indices $\mathbb{N}$, such that two indices, say i and j, belong to the same block of the partition if and only if the points U_i and U_j belong to the same fragment of the object. By the law of large numbers, we see that the masses of the fragments can be recovered as the asymptotic frequencies of the blocks of the partition. Roughly, the fundamental point in Kingman's coding is that it translates the process to a discrete state-space. We refer to Pitman [24] for an important application of these ideas to a coalescent setting.

In the second part of this work, we shall present further applications of Kingman's idea to homogeneous fragmentations. In particular, we will show that spatial discretization is especially well-suited to adapt the conceptual method of probability tilting introduced by Lyons, Pemantle and Peres (see e.g. [22]) to homogeneous fragmentations, which yields also some interesting limit theorems.

2 Time discretization for ranked fragmentations

This section is devoted to the presentation of some applications of time-discretization to the asymptotic behaviour of ranked fragmentations as time tends to ∞. We shall first introduce some notation and definition; then we shall merely translate results of Biggins on branching random walks in continuous time in the special case when the so-called dislocation measure of the fragmentation is finite. Finally, we shall extend the preceding results to the case when the dislocation measure is infinite, and also derive from the extension of a result of Rouault sharp estimates for the probability of presence of abnormally large fragments.

2.1 Some notation and definition

Throughout this paper, we will work with the space of numerical sequences

$$\mathcal{S} := \left\{ \mathbf{s} = (s_1, \dots) : s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \sum_1^\infty s_i \leq 1 \right\}$$

endowed with the uniform distance, which is a compact set. A configuration $s \in \mathcal{S}$ should be thought of as the ranked masses of the fragments resulting from the split of some object with unit total mass.

We consider a family of Feller processes $X = (X_t, t \geq 0)$ with values in $\mathcal{S}$ and càdlàg paths. For every $a \in [0,1]$, we let $\mathbb{P}_a$ denote the law of X with initial distribution $(a, 0, \dots)$ (i.e. the process starts from a single fragment with mass a). We say that X is a (ranked) *homogeneous fragmentation* if the following two properties hold:

• (Homogeneity property) For every $a \in [0,1]$, the law of aX under $\mathbb{P}_1$ is $\mathbb{P}_a$.

• (Fragmentation property) For every $\mathbf{s} = (s_1, \dots) \in \mathcal{S}$, the process started from $X(0) = \mathbf{s}$ can be obtained as follows. Consider $X^{(1)}, \dots$ a sequence of independent processes with respective laws $\mathbb{P}_{s_1}, \dots$, and for every $t \geq 0$, let $\hat{X}(t)$ be the random sequence obtained by ranking in decreasing order the terms of the random sequences $X^{(1)}(t), \dots$. Then $\hat{X}$ has the law of X started from $\mathbf{s}$.

It has been shown in [3] and [4] that homogeneous fragmentations result from the combination of two different phenomena: a continuous erosion and sudden dislocations. The erosion is a continuous deterministic mechanism; dealing with erosion is straightforward, and therefore we will only consider homogeneous fragmentations with no erosion in this work.

The dislocations occur randomly and can be viewed as the jump-component of the process. Roughly speaking, their distribution can be characterized by a measure ν on $\mathcal{S}$, called the *dislocation measure*. Informally ν specifies the rates at which a unit mass splits; see the forthcoming Section 2.2. It has to fulfil the conditions $\nu(\{(1, 0, \dots)\}) = 0$ and

$$\int_{\mathcal{S}} (1 - s_1)\, \nu(d\mathbf{s}) \, < \, \infty \, . \tag{1}$$

More precisely, (1) is the necessary and sufficient condition for a measure ν on $\mathcal{S}$ to be the dislocation measure of some homogeneous fragmentation (see [3] and [4]). We shall assume throughout this work that

$$\nu\left(\left\{\mathbf{s}\in\mathcal{S}:\sum_{i=1}^{\infty}s_i<1\right\}\right)\;=\;0\,, \tag{2}$$

which means that no mass is lost when a sudden dislocation occurs, and more precisely, entails that the total mass is a conserved quantity for the fragmentation process (i.e. $\sum_{i=1}^{\infty}X_i(t)=1$ for all $t\geq 0$, $\mathbb{P}_1$-a.s.). In the sequel, we shall also implicitly exclude the trivial case when $\nu\equiv 0$.

Given a real number $r>0$, we say that a dislocation measure ν is r-geometric if ν is finite and is carried by the subspace of configurations $\mathbf{s}=(s_1,\ldots)\in\mathcal{S}$ such that $s_i\in\{r^{-n},n\in\mathbb{N}\}$. This holds if and only if $\mathbb{P}_1(X_i(t)\in\{r^{-n},n\in\mathbb{N}\}$ for every $i\in\mathbb{N})=1$ for all $t\geq 0$. We say that a dislocation measure is *non-geometric* if it is not r-geometric for any $r>0$.

We now introduce analytic quantities defined in terms of ν which will have an important role in this work. First, we set

$$\underline{p}\;:=\;\inf\left\{p\in\mathbb{R}:\int_{\mathcal{S}}\sum_{i=2}^{\infty}s_i^{p+1}\nu(d\mathbf{s})<\infty\right\}\,,$$

and then for every $q>\underline{p}$

$$\Phi(q)\;=\;\int_{\mathcal{S}}\left(1-\sum_{i=1}^{\infty}s_i^{q+1}\right)\nu(d\mathbf{s})\,. \tag{3}$$

The function Φ is a concave analytic increasing function; it is easy to see (cf. Lemma 1 in [6]) that the equation

$$\Phi(q)\;=\;(q+1)\Phi'(q)\,,\qquad q>\underline{p}$$

has a unique solution, which we denote by $\bar{p}$. More precisely, we then have

$$\Phi(q)-(q+1)\Phi'(q)\;<\;0\quad\Longleftrightarrow\quad q\in]\underline{p},\bar{p}[\,, \tag{4}$$

and

$$\text{the map } q\to\Phi(q)/(q+1) \text{ increases on }]\underline{p},\bar{p}[\text{ and decreases on }]\bar{p},\infty[. \tag{5}$$

To start with, we consider the simple sub-family of fragmentation processes with a finite dislocation measure. These can be reduced to continuous time branching random walks, and we specify some important results of Biggins [9] in this setting. Then we shall investigate the case when the dislocation measure is infinite by time discretization.

2.2 The case when the dislocation measure is finite

Throughout this section, we assume that the dislocation measure ν is finite. It is easy to construct a fragmentation process $X=(X(t),t\geq 0)$ with dislocation measure ν. Let the process start, say from the state $\mathbf{1}:=(1,0,\ldots)$, and stay there for an exponential time with parameter $\nu(\mathcal{S})$. Then the process jumps independently of the waiting time to some random state in $\mathcal{S}$ distributed according to the probability measure $\nu(\cdot)/\nu(\mathcal{S})$. After this first split, each fragment has a similar evolution, independently of the other fragments. In words, a fragment with mass $x\in]0,1[$ breaks after some exponential time with parameter $\nu(\mathcal{S})$, and produces a random sequence of smaller fragments, say xS, where S is a random variable in $\mathcal{S}$ with law $\nu(\cdot)/\nu(\mathcal{S})$.

Plainly, the empirical measure of the logarithms of the fragments

$$Z^{(t)} := \sum_{i=1}^{\infty} \delta_{\log X_i(t)}, \qquad t \geq 0 \tag{6}$$

can be viewed as a branching random walk in continuous-time; see Uchiyama [27], Biggins [9], Kyprianou [17], ... In this direction, let us identify two key quantities related to branching random walks in the fragmentation setting.

First, it is easy to see by an application of the Markov property at the first splitting (i.e. branching) time, that the Laplace transform of the intensity of the point process $Z^{(t)}$ is given for $\theta > \underline{p} + 1$ by

$$m(\theta)^t := \mathbb{E}\left(\int_{\mathbb{R}} \mathrm{e}^{\theta x} Z^{(t)}(dx)\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} X_i(t)^{\theta}\right) = \exp(-t\Phi(\theta - 1)). \tag{7}$$

Second, there is also the identification for the so-called additive martingale

$$W^{(t)}(\theta) := m(\theta)^{-t} \int_{\mathbb{R}} \mathrm{e}^{\theta x} Z^{(t)}(dx) = \exp(t\Phi(\theta - 1)) \sum_{i=1}^{\infty} X_i(t)^{\theta}.$$

These observations allow us to apply Theorem 6 of Biggins [9] (see also [8]), and we now state:

Proposition 1 *Assume that the dislocation measure ν is finite. Then for every $p > \underline{p}$, the process*

$$M(p,t) := W^{(t)}(p+1) = \exp(t\Phi(p)) \sum_{i=1}^{\infty} X_i^{p+1}(t), \qquad t \geq 0$$

is a martingale with càdlàg paths. This martingale converges uniformly on any compact subset of $]\underline{p}, \bar{p}[$, almost surely and in mean, as $t \to \infty$.

We also point out that for $p \geq \bar{p}$, the result of Biggins [8] entails that $\lim_{n\to\infty} M(p,n) = 0$ a.s., and thus by the convergence theorem of càdlàg nonnegative martingales, it holds that $\lim_{t\to\infty} M(p,t) = 0$ a.s.

Proof: Fix some compact interval $[a,b] \subset]\underline{p}, \bar{p}[$, and recall that the event that

$$\sum_{i=1}^{\infty} X_i^{a+1}(t) < \infty \qquad \text{for all } t \geq 0 \tag{8}$$

has probability one. In the sequel, we shall always work on this event. We consider the random continuous function on $[a,b]$

$$M(t) : p \to \exp(t\Phi(p)) \sum_{i=1}^{\infty} X_i^{p+1}(t),$$

which defines a martingale with values in the Banach space $\mathcal{C}([a,b],\mathbb{R})$.

Next, since $\sum_{i=1}^{\infty} s_i = 1$ for ν-a.e. $\mathbf{s} \in \mathcal{S}$, observe that, for every $\theta > \underline{p}+1$, we have $\gamma(\theta-1) > \underline{p}$ for some $\gamma > 1$ and then Jensen's inequality implies that

$$\int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} s_i^{\theta} \right)^{\gamma} \nu(d\mathbf{s}) \;\leq\; \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} s_i^{\theta\gamma+1-\gamma} \right) \nu(d\mathbf{s}) \;<\; \infty\,.$$

On the other hand, thanks to (5), we get that for every $\theta \in]\underline{p}+1, \bar{p}+1[$, there exists $\alpha \in]1,\gamma[$ such that

$$m(\alpha\theta) < m(\theta)^{\alpha}\,.$$

By the argument used in the proof of Theorem 6 in Biggins [9] (see also Remark (1) at the end of the present proof), all that needs to be checked is the martingale $(M(t), t \geq 0)$ has right-continuous paths. We stress that the argument we give does not rely on the assumption of finiteness for the dislocation measure.

In this direction, it is convenient to use an interval representation of the fragmentation (see [5]). Specifically, we consider the space of open subsets of the unit interval, endowed with the metric induced by the Hausdorff distance for the complementary closed set. One can construct a right-continuous family $(\Theta(t), t \geq 0)$ of random open subsets of the unit interval such that for each $t \geq 0$, $X(t)$ is the ranked sequence of the lengths of the interval components of $\Theta(t)$, and $\Theta(t') \subseteq \Theta(t)$ whenever $t \leq t'$. Recall also that the assumption (2) ensures that each $\Theta(t)$ has full Lebesgue measure a.s.

For every $x \in]0,1[$ and $q \in [a,b]$, write $f_{t,q}(x) = |I_x(t)|^q$, where $I_x(t)$ denotes the interval component of $\Theta(t)$ that contains x and $|I_x(t)|$ its length. In this setting, we thus have

$$\sum_{i=1}^{\infty} X_i^{q+1}(t) \;=\; \int_0^1 f_{t,q}(x)dx\,.$$

For every $t_0 \geq 0$ and $x \in \Theta(t_0)$, we get from the right-continuity of $(\Theta(t), t \geq 0)$ that $|I_x(t)|$ increases to $|I_x(t_0)|$ as t decreases to t_0. The obvious upper-bounds

$$f_{t,q}(x) \;\leq\; f_{t',a}(x)\,, \qquad t \leq t' \text{ and } x \in]0,1[\;\; q \leq 0$$

$$f_{t,q}(x) \;\leq\; f_{t_0,a}(x)\,, \qquad t \geq t_0 \text{ and } x \in]0,1[\;\; q \geq 0$$

combined with (8) enable us to apply the theorem of dominated convergence, and hence

$$\lim_{t \to t_0+} \int_0^1 f_{t,q}(x)dx \;=\; \int_0^1 f_{t_0,q}(x)dx\,.$$

This shows that with probability one, the real-valued martingales $M(q,\cdot)$ have right-continuous paths for all $q \in [a,b]$.

To conclude, we observe that the random function $q \to \sum_{i=1}^{\infty} X_i^{q+1}(t)$ is continuous and decreases as q increases. An appeal to Dini's Theorem now shows that $(M(t), t \geq 0)$ is right-continuous at all $t \geq 0$, with probability one. □

Remarks. (1) It seems that the discretization argument in the proof of the almost sure convergence in Theorem 6 in [9] might require a further explanation. Indeed, it is observed there that $W^{(n\delta)}$ converges a.s. when $n \to \infty$ through integers for any $\delta > 0$, and then claimed that this implies the a.s. convergence of $W^{(t)}$ when $t \to \infty$ through the rationals. The latter assertion does not look obvious, so we propose a slightly different argument. One works with the martingale $W^{(t)}$ with values in the space of continuous functions on some compact space, endowed with the supremum norm, $||\cdot||$. We know that this martingale converges in mean as $t \to \infty$. The norm is a convex map, therefore for every integer n, the process $||W^{(t+n)} - W^{(n)}||$ is a nonnegative submartingale with regular paths. Doob's maximal inequality now entails that for every $\varepsilon > 0$

$$\mathbb{P}\left(\exists t > 0 : ||W^{(t+n)} - W^{(n)}|| > \varepsilon\right) \leq \varepsilon^{-1} \sup_{t \geq 0} \mathbb{E}\left(||W^{(t+n)} - W^{(n)}||\right),$$

and the right-hand side converges to 0 as $n \to \infty$. An application of the Borel-Cantelli lemma now completes the proof of the almost sure convergence of $W^{(t)}$ as $t \to \infty$.

(2) We also mention that Proposition 1 can be extended to complex numbers p with $\underline{p} < \Re p < \bar{p}$. The key point is to check that the martingale $p \to M(p,t)$ ($t \geq 0$), viewed as a process with values in the space $\mathcal{C}(K, \mathbb{C})$ for some compact set $K \subset \{z : \underline{p} < \Re p < \bar{p}\}$, has right-continuous paths. We know from the proof of Proposition 1 that the latter holds when K is a real segment, and the general case follows easily.

Next, we derive an important consequence of the preceding analysis concerning almost sure large deviations for the empirical measure; see Corollary 4 and the discussion on page 150 in Biggins [9].

Corollary 2 *Assume that the dislocation measure ν is finite and non-geometric. For $p \in]\underline{p}, \bar{p}[$, let $M(p,\infty)$ be the terminal value of the uniformly integrable martingale $M(p,\cdot)$. If $f : \mathbb{R} \to \mathbb{R}$ is a function with compact support which is directly Riemann integrable, then*

$$\lim_{t\to\infty} \sqrt{t}\, \mathrm{e}^{-t((p+1)\Phi'(p) - \Phi(p))} \int_{\mathbb{R}} f(t\Phi'(p) + y) Z^{(t)}(dy) = \frac{M(p,\infty)}{\sqrt{2\pi|\Phi''(p)|}} \int_{-\infty}^{\infty} f(y) \mathrm{e}^{-(p+1)y} dy\,.$$

uniformly for p in compact subsets of $]\underline{p}, \bar{p}[$, almost surely.

We point out that this result applied for indicator functions of bounded intervals gives a sharp large deviation statement that extends Corollary 2 of [6].

2.3 The case when the dislocation measure is infinite

We now drop the assumption of finiteness of the dislocation measure ν, and merely assume that (1) holds. As above we associate to the fragmentation X the empirical measures $Z^{(t)}$ defined in (6); note that when $\nu(\mathcal{S}) = \infty$, the process $(Z^{(t)}, t \geq 0)$ is no longer of the kind considered by Uchiyama [27].

Theorem 3 *Proposition 1 and Corollary 2 hold when one relaxes the requirement of finiteness of the dislocation measure ν and merely assumes (1).*

Proof: The restriction of the process of empirical distribution to integers times $(Z^{(n)}, n = 0, 1, \ldots)$ is a branching random walk; we aim at applying results of Biggins [9] in this setting.

The main difficulty is that the distribution of $Z^{(1)}$ is not explicitly known in terms of the dislocation measure ν. However, it is known that (7) still holds; see e.g. the identity (6) in [6]. Moreover, the proof of Theorem 2 in [6] shows that for every $p > \underline{p}$, there exists some $\gamma > 1$ such that for every $t \geq 0$

$$\mathbb{E}\left(\left(\sum_{i=1}^{\infty} X_i^{p+1}(t)\right)^{\gamma}\right) < \infty \,. \tag{9}$$

Theorem 2 of Biggins [9] now shows that the conclusions of Proposition 1 hold provided that $t \to \infty$ through integers. We can complete the argument as in Theorem 6 of Biggins [9] (using also the proof of Proposition 1 in the present paper and the remark thereafter). The extension of Corollary 2 is proven by adapting the arguments of Biggins [9] on page 150. □

In a different direction, one can also use the skeleton method to estimate the probability of presence of abnormally large fragments as time goes to infinity. Indeed, a similar problem has been solved for branching random walks, see [25] and the references therein. Informally in the so-called sub-critical region, the probability of presence of particles is of the same order as the mean number of particles in that region, and the asymptotic behaviour of the latter can be derived from the local central limit theorem. This incites us to fix two real numbers $\alpha < \beta$ and to introduce for every $t \geq 0$ and $x \in \mathbb{R}$ the notation:

$$U(t,x) \;:=\; \mathbb{P}(Z^{(t)}([x+\alpha, x+\beta]) > 0) \tag{10}$$

$$V(t,x) \;:=\; \mathbb{E}\left(Z^{(t)}([x+\alpha, x+\beta])\right) \,. \tag{11}$$

Theorem 4 *Assume that the dislocation measure ν is non-geometric.*

(i) *If $p > \underline{p}$, we have*

$$\lim_{t\to\infty} \sqrt{t}\, \mathrm{e}^{-t((p+1)\Phi'(p)-\Phi(p))} V(t, -t\Phi'(p)) \;=\; \frac{1}{\sqrt{2\pi|\Phi''(p)|}}(p+1)^{-1}\left(\mathrm{e}^{-(p+1)\alpha} - \mathrm{e}^{-(p+1)\beta}\right) \,.$$

(ii) *If $p > \bar{p}$, there exists a positive finite constant K_p such that*

$$\lim_{t\to\infty} \frac{U(t, -t\Phi'(p))}{V(t, -t\Phi'(p))} = K_p \,.$$

We point out that in the range $p \in]\underline{p}, \bar{p}[$, (i) is the counterpart in mean of the result of Corollary 2, when $f = \mathbf{1}_{[\alpha,\beta]}$. It implies that the convergence there holds also in $L^1(\mathbb{P})$, thanks to Scheffé's theorem.

Proof: The proof relies on a result of Rouault [25] for branching random walks, which has been extended under more general conditions in [7], and that we take here for granted. Recall also (9), which ensures that the assumptions of [7] are satisfied.

We apply the skeleton method. Let $h > 0$ be a time mesh; the fragmentation process observed at times nh ($n \in \mathbb{N}$) yields a branching random walk. Write

$$\widehat{Z^{(h)}}(\theta) := \int_{\mathbb{R}} \mathrm{e}^{\theta x} Z^{(h)}(dx) = \sum_i X_i(h)^\theta$$

and

$$\Lambda_h(\theta) := \log \mathbb{E}\left(\widehat{Z^{(h)}}(\theta)\right) = -h\Phi(\theta - 1)\,.$$

In the case $h = 1$, we write for simplicity $\Lambda = \Lambda_1$ and set

$$a = -\Phi'(p)\ ,\ \sigma_p^2 = -\Phi''(p)\ ,\ \Lambda^*(a) = \Phi(p) - (p+1)\Phi'(p)\,.$$

For an arbitrary mesh $h > 0$, we define by scaling

$$\Lambda_h^*(x) = h\Lambda^*(x/h)\,.$$

It is immediately checked that if θ solves $a = \Lambda_h'(\theta)$, then $\Lambda_h^*(a) = \theta\Lambda_h'(\theta) - \Lambda_h(\theta)$. Applying Theorem 2 in [7], we get first

$$\lim_n \sigma_p \sqrt{2\pi nh}\ \mathrm{e}^{nh\Lambda^*(a)} V(nh, anh) = (p+1)^{-1}\left(\mathrm{e}^{-(p+1)\alpha} - \mathrm{e}^{-(p+1)\beta}\right), \tag{12}$$

and then that, if $\Lambda_h^*(a) > 0$ the limit

$$\lim_n \frac{U(nh, anh)}{V(nh, anh)} =: K_p^{(h)}$$

exists for each $h > 0$ and is positive. Now, it is easy to see that the functions

$$t \mapsto \sigma_p \sqrt{2\pi t}\ \mathrm{e}^{t\Lambda^*(a)} V(t, at) \ \text{ and } \ t \mapsto \frac{U(t, at)}{V(t, at)} \tag{13}$$

are continuous. We apply the Croft-Kingman lemma ([1] A 9.1 p.438, see also [14]). Both limits exist when $t \to \infty$. In the first case, it is of course the right hand side of (12). In the second case, it is any $K_p^{(h)}$ since they are all equal. □

3 Spatial discretization and fragmentation of partitions

This section is devoted to another useful discretization technique which has been sketched in the Introduction and that we now recall for convenience. We may suppose that we are given a sequence of i.i.d. random points $U_1, \ldots$ which are picked according to the mass distribution of the object. These random points are assumed to be independent of the fragmentation process. One then looks at the fragmentation process as the evolution as time t passes of the random partition $\Pi(t)$ of $\mathbb{N}$ which is given as follows. Two indices i and j are in the same block of $\Pi(t)$ if and only if the points U_i and U_j belong to the same fragment of the object at time t. Plainly, the random partition gets finer and finer as time passes.

We first present in Section 3.1 the necessary background on partitions of $\mathbb{N}$, and then in Section 3.2 the Poissonian construction of homogeneous fragmentations and the connection with subordinators, following closely [4]. Section 3.3 is devoted to the study of probability tilting based on additive martingales, adapting to the random partition setting the so-called conceptual method of Lyons et al. [22, 21]. Finally, as an example of application, we investigate in Section 3.4 the convergence of the so-called derivative martingale.

3.1 Preliminaries

A *partition* of $\mathbb{N} = \{1, \ldots\}$ is a sequence $\pi = (\pi_1, \pi_2, \ldots)$ of disjoint subsets, called *blocks*, such that $\bigcup \pi_i = \mathbb{N}$. The blocks π_i of a partition are enumerated in the increasing order of their least element, i.e. $\min \pi_i \leq \min \pi_j$ for $i \leq j$, with the convention that $\min \emptyset = \infty$. If π and π' are two partitions of $\mathbb{N}$, we say that π is finer than π' if every block of π is contained into some block of π'.

For every block $B \subseteq \mathbb{N}$, we denote by $\pi_{|B}$ the partition of B induced by π and an obvious restriction. For every integer k, the block $\{1, \ldots, k\}$ is denoted by $[k]$. A partition π is entirely determined by the sequence of its restrictions $\left(\pi_{|[k]}, k \in \mathbb{N}\right)$, and conversely, if for every $k \in \mathbb{N}$, γ_k is a partition of $[k]$ such that the restriction of γ_{k+1} to $[k]$ coincides with γ_k (this will be referred to the *compatibility property* in the sequel), then there exists a unique partition $\pi \in \mathcal{P}$ such that $\pi_{|[k]} = \gamma_k$ for every $k \in \mathbb{N}$.

The space of partitions of $\mathbb{N}$ is denoted by $\mathcal{P}$ and endowed with the hyper-distance

$$\mathrm{dist}(\pi, \pi') := 1/\max\left\{k \in \mathbb{N} : \pi_{|[k]} = \pi'_{|[k]}\right\} ,$$

with the convention $1/\max \mathbb{N} := 0$. This makes $\mathcal{P}$ compact.

One says that a block $B \subseteq \mathbb{N}$ has an *asymptotic frequency* if the limit

$$|B| := \lim_{n \to \infty} n^{-1} \mathrm{Card}(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, \ldots)$, and then $|\pi|^{\downarrow} = (|\pi|_1^{\downarrow}, \ldots) \in \mathcal{S}$ for the decreasing rearrangement[1] of the sequence $|\pi|$. In the case when some block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}$. This defines a natural map $\pi \to |\pi|^{\downarrow}$ from $\mathcal{P}$ to $\mathcal{S} \cup \{\partial\}$ which is not continuous.

We call *nested partitions* a collection $\Pi = (\Pi(t), t \geq 0)$ of partitions of $\mathbb{N}$ such that $\Pi(t)$ is finer than $\Pi(t')$ when $t' \leq t$. There is a simple procedure for the construction of a large family of nested partitions which we now describe and will use throughout the rest of this section.

We call *discrete point measure* on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ any measure ω which can be expressed in the form

$$\omega = \sum_{(t,\pi,k) \in \mathcal{D}}^{\infty} \delta_{(t,\pi,k)}$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that the following two requirements hold:

- For every $t \in \mathbb{R}$, $\omega(\{t\} \times \mathcal{P} \times \mathbb{N}) = 0$ or 1.
- For every real number $t' \geq 0$ and integer $n \geq 1$

$$\mathrm{Card}\left\{(t, \pi, k) \in \mathcal{D} : t \leq t', \pi_{|[n]} \neq \mathrm{trivial}(n), k \leq n\right\} < \infty ,$$

[1] Ranking the asymptotic frequencies of the blocks of π in the decreasing order is just a simple procedure to forget the labels of these blocks. In other words, we want to consider the family of the asymptotic frequencies without keeping the additional information provided by the way blocks are labelled.

where trivial$(n) = ([n], \emptyset, \emptyset, \ldots)$ stands for the partition of $[n]$ which has a single non empty block[2].

Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we may construct nested partitions $\Pi = (\Pi(t), t \geq 0)$ as follows: Fix $n \in \mathbb{N}$; the assumption that the point measure ω is discrete enables us to construct a step-path $(\Pi(t,n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom of ω, say (t, π, k), such that $\pi_{|[n]} \neq \text{trivial}(n)$ and $k \leq n$. In that case, $\Pi(t,n)$ is the partition obtained by replacing the k-th block of $\Pi(t-, n)$, viz. $\Pi_k(t-, n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the other blocks unchanged. Now it is immediate from this construction that for each time $t \geq 0$, the sequence $(\Pi(t,n), n \in \mathbb{N})$ is compatible, and hence there exists a unique partition $\Pi(t)$ such that $\Pi(t)_{|[n]} = \Pi(t,n)$ for each $n \in \mathbb{N}$.

3.2 Poisson measures, homogeneous fragmentations, and subordinators

We denote the space of discrete point measures on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ by Ω, and the sigma-field generated by the restriction to $[0,t] \times \mathcal{P} \times \mathbb{N}$ by $\mathcal{G}(t)$. So $(\mathcal{G}(t))_{t \geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $(\mathcal{G}(t))_{t \geq 0}$-adapted. We shall also need to consider the sigma-field $\mathcal{F}(t)$ generated by the decreasing rearrangement $|\Pi(r)|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$ for $r \leq t$, and $(\mathcal{F}(t))_{t \geq 0}$ is a sub-filtration of $(\mathcal{G}(t))_{t \geq 0}$.

Now consider a dislocation measure ν, i.e. a measure on $\mathcal{S}$ which fulfils the requirements of Section 2.1. According to Theorem 2 in [4], there exists a unique measure μ on $\mathcal{P}$ which is *exchangeable* (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that ν is the image of μ by the map $\pi \to |\pi|^{\downarrow}$. An important fact which stems from exchangeability, is that the distribution of the asymptotic frequency of the first block $|\pi_1|$ under the measure μ is that of a size-biased picked term from the ranked sequence s under ν. In other words, there is the identity

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(d\pi) \; = \; \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i f(s_i)\nu(d\mathbf{s}) \,, \tag{14}$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function with $f(0) = 0$.

Let $\mathbb{P}$ be the probability measure on Ω corresponding to the law of a Poisson point measure with intensity $dt \otimes \mu \otimes \#$, where $\#$ denotes the counting measure on $\mathbb{N}$. The assumption (1) on the dislocation measure ν ensures that ω is a discrete point measure $\mathbb{P}$-a.s. The nested partitions $(\Pi(t), t \geq 0)$ constructed above from ω now form a Markov process; see Section 3 in [4]. More precisely the Markov property is essentially a variation of the branching property; it can be stated as follows. For every $t, t' \geq 0$, the conditional distribution of $\Pi(t + t')$ given $\mathcal{G}(t)$ is the same as that of the random partition of $\mathbb{N}$ induced by the restrictions $\Pi^{(1)}(t')_{|B_1}, \Pi^{(2)}(t')_{|B_2}, \ldots$, where $\Pi^{(1)}, \ldots$ are independent copies of Π and $(B_1, \ldots) = \Pi(t)$ is the sequence of blocks of $\Pi(t)$. In the terminology of [4], we say that $\Pi = (\Pi(t), t \geq 0)$ is a (partition valued) *homogeneous fragmentation* under $\mathbb{P}$.

Another crucial fact is that the partitions $(\Pi(t), t \geq 0)$ are exchangeable under $\mathbb{P}$, i.e. their

[2]Roughly, trivial(n) plays the role of a neutral element in the space of partitions of $[n]$.

distribution is invariant under the action of finite permutations on $\mathbb{N}$; see Section 3 in [4]. It follows from a celebrated theorem of Kingman [15] that $\mathbb{P}$-a.s., $\Pi(t)$ has asymptotic frequencies for all $t \geq 0$; cf. Theorem 3(i) in [4]. The process of ranked asymptotic frequencies $|\Pi|^{\downarrow} := X$ is a Markov process with values in $\mathcal{S}$; it provides a version of the ranked fragmentation which we considered in Section 2; cf. [3].

The *tagged fragment* is the fragment of the object that contains the first tagged point U_1, i.e. which corresponds to the first block $\Pi_1(\cdot)$. The process $|\Pi_1(\cdot)|$ of the asymptotic frequencies of the first block and its logarithm,

$$\xi(t) := -\log|\Pi_1(t)|, \qquad t \geq 0$$

will have a special role in this study. A crucial point is that under $\mathbb{P}$, $\xi = (\xi_t, t \geq 0)$ is a subordinator with Laplace exponent Φ, which is given by (3); cf. Theorem 3(ii) in [6]. This means that $(\xi(t), t \geq 0)$ is a càdlàg process with independent and stationary increments, and the Laplace transform of its one-dimensional distribution is given by the identity

$$\mathbb{E}\left(\exp(-q\xi(t)\right) = \exp(-t\Phi(q)), \qquad q > \underline{p}.$$

More precisely, let us denote by $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure ω to the fiber $[0,t] \times \mathcal{P} \times \{1\}$. So $(\mathcal{G}_1(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$, and the first block of $\Pi(t)$, $\Pi_1(t)$, and a fortiori its asymptotic frequency $\mathrm{e}^{-\xi_t}$, are $\mathcal{G}_1(t)$-measurable. Let $\mathcal{D}_1 \subseteq [0,\infty[$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\} \times \mathcal{P} \times \{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$. The construction of the nested partitions from the discrete point measure yields the identity

$$\exp\left(-\xi_t\right) = |\Pi_1(t)| = \prod_{r \in \mathcal{D}_1 \cap [0,t]} |\pi_1(r)|, \tag{15}$$

for all $t \geq 0$, a.s. under $\mathbb{P}$; see e.g. the first remark at the end of Section 5 in [4]. Observe that taking logarithm turns the identity (15) into the Lévy-Itô decomposition for subordinators.

Finally, the conditional distribution of the size of the tagged fragment, $|\Pi_1(t)| = \mathrm{e}^{-\xi(t)}$, given $\mathcal{F}(t)$ (the sigma-field generated by the ranked asymptotic frequencies) is that of a size-biased sample from the ranked sequence $|\Pi(t)|^{\downarrow}$. In other words, we have

$$\mathbb{E}\left(f(\exp(-\xi(t))\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} |\Pi_i(t)| f(|\Pi_i(t)|)\right) = \mathbb{E}\left(\sum_{j=1}^{\infty} X_j(t) f(X_j(t))\right)$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function with $f(0) = 0$. More generally, exchangeability ensures that for every $t \geq 0$, the sequence $|\Pi(t)|$ of the asymptotic frequencies is a size-biased reordering of the ranked sequence $X(t) = |\Pi(t)|^{\downarrow}$.

3.3 Additive martingales and tilted probability measures

There are two simple martingales connected to fragmentations for every parameter $p > \underline{p}$: First, a well-known fact for subordinators is that

$$\mathcal{E}(p,t) := \exp(-p\xi(t) + t\Phi(p)) = \mathrm{e}^{t\Phi(p)} |\Pi_1(t)|^p$$

is a positive $(\mathbb{P}, \mathcal{G}(t))$-martingale. Second, when we project $\mathcal{E}(p,t)$ on the sub-filtration $\mathcal{F}(t)$, we recover the additive martingale

$$M(p,t) \;=\; \exp(t\Phi(p)) \sum_{i=1}^{\infty} |\Pi_i(t)|^{p+1} \;=\; \exp(t\Phi(p)) \sum_{j=1}^{\infty} X_j^{p+1} \,.$$

We point out that, more precisely, $M(p,\cdot)$ is a $(\mathbb{P}, \mathcal{G}(t))$-martingale which is adapted to the sub-filtration $\mathcal{F}(t)$.

Following the genuine method of Lyons, Pemantle and Peres (see e.g. [22]), we introduce the *tilted probability measure* $\mathbb{P}^{(p)}$ on the space of discrete point measures Ω endowed with the filtration $(\mathcal{G}(t))_{t\geq 0}$ by

$$d\mathbb{P}^{(p)}_{|\mathcal{G}(t)} \;=\; \mathcal{E}(p,t)\, d\mathbb{P}_{|\mathcal{G}(t)} \,. \tag{16}$$

Observe that projections on the sub-filtration $\mathcal{F}(t)$ give the identity

$$d\mathbb{P}^{(p)}_{|\mathcal{F}(t)} \;=\; M(p,t)\, d\mathbb{P}_{|\mathcal{F}(t)} \,. \tag{17}$$

The effect of the change of probability is easy to describe, both at the level of the tagged fragment and that of the discrete point measure.

Proposition 5 (i) *Under $\mathbb{P}^{(p)}$, the process $\xi_t \;=\; -\log|\Pi_1(t)|$ is a subordinator with Laplace exponent*

$$\Phi^{(p)}(q) \;:=\; \Phi(p+q) - \Phi(p)\,, \qquad q > \underline{p} - p\,.$$

(ii) *Under $\mathbb{P}^{(p)}$, the discrete point measure ω is Poissonian. More precisely:*

- *The restriction of ω to $\mathbb{R}_+ \times \mathcal{P} \times \{2, 3, \ldots\}$ has the same distribution as under $\mathbb{P}$ and is independent of the restriction to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$.*
- *In the notation of Section 3.1, the family $\{(r, \pi(r)), r \in \mathcal{D}_1\}$ is that of the atoms of a Poisson random measure on $\mathbb{R}_+ \times \mathcal{P}$ with intensity $dr \otimes \mu^{(p)}$, where*

$$\mu^{(p)}(d\pi) \;=\; |\pi_1|^p \mu(d\pi)\,.$$

Proof: The first assertion stems from the classical Esscher transform; see for instance Example 33.15 in Sato [26]. The description of the law of the discrete point measure under $\mathbb{P}^{(p)}$ is easily seen from the formula (15), and classical properties of the exponential tilting for Poisson random measures. □

We stress that the tilting only affects the distribution of the discrete point measure on the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$. In this direction, it is interesting to use Proposition 5(ii) and compare the evolution of the ranked-fragmentation $X = |\Pi(\cdot)|^{\downarrow}$ under $\mathbb{P}^{(p)}$ with the evolution under $\mathbb{P}$. For the sake of simplicity, we again suppose here that the dislocation measure ν is finite, so the evolution of the ranked fragmentation under $\mathbb{P}$ is described in section 2.2.

First, we observe that by absolute continuity, the random partition $\Pi(t)$ obtained by evaluating the nested partitions at time t, possesses asymptotic frequencies a.s. under the tilted probability $\mathbb{P}^{(p)}$. The first block $\Pi_1(\cdot)$ has a special role in the definition of the tilted probability $\mathbb{P}^{(p)}$, and cannot be recovered from the ranked sequence $|\Pi(\cdot)|^{\downarrow}$ alone. Let us call *marked* the unique particle (i.e. asymptotic frequency) at time t corresponding to $\Pi_1(t)$ and *unmarked* the other ones. Observe that under $\mathbb{P}$, the tagged fragment coincides with the marked particle, and that under $\mathbb{P}^{(p)}$, the unmarked particles follow the same evolution as under $\mathbb{P}$, i.e. they split according to ν, independently of the others, and only produce unmarked particles. Under $\mathbb{P}^{(p)}$, the marked particle splits independently of the other particles, but with a different rate, namely

$$\nu^{(p)}(ds) := \left(\sum_{i=1}^{\infty} s_i^{p+1}\right) \nu(d\mathbf{s}) .$$

Indeed, we deduce from Proposition 5(ii) that $\nu^{(p)}$ is the image of the intensity measure $\mu^{(p)}$ by the map $\pi \to |\pi|^{\downarrow}$, and since under μ, $|\pi_1|$ can be viewed as a size-biased pick from the ranked sequence $|\pi|^{\downarrow}$ (recall the identity (14)), this yields the formula above. The "new" marked particle is picked at random amongst the particles produced by the splitting of the "old" marked particle as follows: Let x denote the mass of the old marked particle and xs the ranked sequence of the masses of the particles produced after the splitting, where $s = (s_1, \ldots) \in \mathcal{S}$. Then the probability that the new marked particle has mass xs_j equals $s_j^{p+1} / \sum_{i=1}^{\infty} s_i^{p+1}$. In short, the marked particle $\Pi_1(\cdot)$ can be viewed as a canonic analog in the fragmentation setting of the so-called *spine* in the branching random walk framework.

It may be interesting to point at the following connection with the so-called *thinning* of discrete point measures. Recall that, given some metric space A, a discrete point measure μ on A with atoms $x_1, \ldots$ (i.e. $\mu = \sum_{i=1}^{\infty} \delta_{x_i}$), and a measurable function $f : A \to [0,1]$, an f-thinning of μ is the random discrete point measure $\mu^{(r)}$ obtained by keeping each atom x of μ with probability $f(x)$, independently of the others. In other words,

$$\mu^{(f)} = \sum_{j=1}^{\infty} \mathbf{1}_{\{\chi_j = 1\}} \delta_{x_j}$$

where $\chi_1, \ldots$ is a sequence of independent Bernoulli variables with $\mathbb{P}(\chi_j = 1) = f(x_j)$. Informally, the following corollary shows that dislocations are less (respectively, more) frequent under $\mathbb{P}^{(p)}$ than under $\mathbb{P}$ for $p > 0$ (respectively for $p < 0$).

Corollary 6 *For every $q > 0$, let f_q be the map on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that $f_q(t, \pi, k) = |\pi_1|^q$ if the partition π possesses asymptotic frequencies and $k = 1$, and $f_q(t, \pi, k) = 0$ otherwise.*

(i) *For every $p > 0$, the image of $\mathbb{P}$ by an f_p-thinning of the discrete point measure ω is $\mathbb{P}^{(p)}$.*

(ii) *For every $p \in]\underline{p}, 0[$, the image of $\mathbb{P}^{(p)}$ by an f_{-p}-thinning of the discrete point measure ω is $\mathbb{P}$.*

Proof: The first statement is immediate from Proposition 5(ii) and properties of thinning (see e.g. Chapter 5 in Kingman [16]). Suppose now that $\underline{p} < p < 0$, and work under $\mathbb{P}$. Let ω' be a random Poisson point measure on the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$ with intensity $dt \otimes (|\pi_1|^p - 1)\mu(d\pi)$, which is independent of ω. By superposition of independent Poisson measures, $\mathbb{P}^{(p)}$ can be identified as the law of $\omega + \omega'$ under $\mathbb{P}$. It follows readily that the original probability measure $\mathbb{P}$ can be recovered from $\mathbb{P}^{(p)}$ by f_{-p}-thinning. $\square$

3.4 The derivative martingale

We end this work by considering the so-called derivative martingale that we now introduce. Recall that $\bar{p} > 0$ is the critical value for the convergence in $L^1(\mathbb{P})$ of the additive martingales. The process

$$\mathcal{E}'(t) := (t\Phi'(\bar{p}) - \xi(t)) \exp(-\bar{p}\xi(t) + t\Phi(\bar{p})) , \qquad t \geq 0$$

is clearly a $(\mathbb{P}, \mathcal{G}(t))$-martingale; its projection on the sub-filtration $(\mathcal{F}(t))_{t\geq 0}$ is a $(\mathbb{P}, \mathcal{F}(t))$ martingale, called the *derivative martingale* and given by

$$M'(t) = \sum_{i=1}^{\infty} (t\Phi'(\bar{p}) + \log(|\Pi_i(t)|)) \exp(t\Phi(\bar{p})) |\Pi_i(t)|^{\bar{p}+1} .$$

We stress that the derivative martingale is not always positive, which contrasts with the case of additive martingales. The idea of considering the derivative martingale at the critical value goes back to Neveu [23] for the branching Brownian motion. For the branching random walk, it has been considered by Kyprianou [18], Liu [20] with the help of a functional equation and by Biggins and Kyprianou [10] with the measure change method.

Proposition 7 (i) *The martingale M' converges $\mathbb{P}$-a.s. to a finite non-positive limit $M'(\infty)$,*
(ii) $\mathbb{E}(M'(\infty)) = -\infty$,
(iii) $\mathbb{P}(M'(\infty) < 0) = 1$.

The proposition could be derived by time discretization from its analog for branching random walks. However, as checking the technical details may be rather involved in this instance, we shall present a direct proof based on probability tilting. This technique is due to Lyons et al. [21]; it can be also applied to establish the uniform integrability of additive martingales (cf. Theorem 3). We also refer to Harris [13] and Kyprianou [19] for related treatments.

Proof of (i): Define for every $i \in \mathbb{N}$ and $s \leq t$, $\beta_{s,t}(i)$ as the unique block of $\Pi(s)$ containing $\Pi_i(t)$. For $a > 0$, let

$$\begin{cases} \Pi_i^{(a)}(t) = \Pi_i(t), & \text{if } |\beta_{s,t}(i)| \leq \exp\{a - s\Phi'(\bar{p})\} \text{ for every } s \leq t; \\ \Pi_i^{(a)}(t) = \emptyset, & \text{otherwise} . \end{cases}$$

The family $\left\{\Pi_i^{(a)}(t) : i \in \mathbb{N}\right\}$ obviously possesses asymptotic frequencies. Moreover, it should be plain that as t varies in $[0, \infty[$, this family of partitions is nested. We denote by $(\mathcal{H}(t))_{t\geq 0}$ the filtration generated by the process of their ranked asymptotic frequencies, so $(\mathcal{H}(t))_{t\geq 0}$ is another sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$.

Because $\Phi'(\bar{p}) = \Phi(\bar{p})/(\bar{p}+1)$ and the martingale $M(\bar{p}, t)$ converges to 0, $\mathbb{P}$-a.s., we have $\sup_{t\geq 0} \left\{\exp(t\Phi'(\bar{p}))|\Pi(t)|_1^{\downarrow}\right\} < \infty$, $\mathbb{P}$-a.s. It follows that

$$\lim_{a\to\infty} \mathbb{P}\left(\Pi_i^{(a)}(t) = \Pi_i(t) \text{ for all } i \in \mathbb{N} \quad \text{and for all } t \geq 0\right) = 1 .$$

Thus, in order to prove the existence of a finite limit for M', it suffices to establish that if

$$M_a(t) := \sum_{i=1}^{\infty} (\log(1/|\Pi_i(t)|) - t\Phi'(\bar{p}) + a) \exp(t\Phi(\bar{p})) |\Pi_i^{(a)}(t)|^{\bar{p}+1} \tag{18}$$

then $\lim_{t\to\infty} M_a(t) =: M_a(\infty)$ exists $\mathbb{P}$-a.s. for every $a > 0$.

From now on, we fix $a > 0$. Since the process $\xi(t) - \Phi'(\bar{p})t$ has no negative jumps,

$$\mathcal{M}_a(t) \;:=\; (\xi(t) + a - t\Phi'(\bar{p}))\exp(-\bar{p}\xi(t) + t\Phi(\bar{p}))\mathbf{1}_{\{t<\zeta_a\}}$$

where $\zeta_a = \inf\{t \geq 0 : \xi(t) < t\Phi'(\bar{p}) - a\}$, can be viewed as a stopped (non-negative) $\mathbb{P}$-martingale. Its projection on the sub-filtration $(\mathcal{H}(t))_{t\geq 0}$ is $M_a(t)$, which therefore is a non-negative $(\mathbb{P}, \mathcal{H}(t))$ martingale, and thus possesses a finite limit as $t \to \infty$, $\mathbb{P}$-a.s. $\square$

Proof of (ii): It is sufficient to show that for all $a > 0$, in the notation above,

$$\liminf_{t\to\infty} M_a(t) < \infty\,, \qquad \mathbb{Q}\text{-a.s.} \tag{19}$$

Indeed (19) entails that the $\mathbb{P}$-martingale M_a is uniformly integrable (see Lyons [22]). Then $\mathbb{E}\left(M_a(t)\right) = a$, which yields $\mathbb{E}(-M'(\infty)) \geq a$ for every $a > 0$. In this direction, we introduce the tilted probability measure $\mathbb{Q}$ on Ω given by

$$d\mathbb{Q}_{|\mathcal{G}(t)} \;=\; a^{-1}\mathcal{M}_a(t)\, d\mathbb{P}_{|\mathcal{G}(t)}\,; \tag{20}$$

so we also have

$$d\mathbb{Q}_{|\mathcal{H}(t)} \;=\; a^{-1}M_a(t)\, d\mathbb{P}_{|\mathcal{H}(t)}\,.$$

The following lemma, which is a simple variation of Proposition 5, lies in the heart of the proof of (19).

Lemma 8 (i) *Under $\mathbb{Q}$, the process*

$$\lambda(t) \;:=\; (\xi(t) + a - t\Phi'(\bar{p}))\,\mathbf{1}_{\{t<\zeta_1\}}\,, \qquad t \geq 0$$

is a centered Lévy process with no negative jumps started from a and conditioned to stay positive forever (see for instance [11]).

(ii) *Under $\mathbb{Q}$, the restriction of ω to $\mathbb{R}_+ \times \mathcal{P} \times \{2, 3, \ldots\}$ has the same distribution as under $\mathbb{P}$ and is independent of the restriction to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$.*

We are now able to complete the proof of Proposition 7(ii).

Proof of (19)**:** It is easily seen from Lemma 8(i) that

$$\inf\{\lambda(t), t \geq 0\} \;>\; 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\log \lambda(t)}{\log t} \;=\; 1/2 \qquad \mathbb{Q}\text{-a.s.} \tag{21}$$

As a consequence,

$$\lim_{t\to\infty} \left(\log\left(1/|\Pi_1(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_1(t)|^{\bar{p}+1} \;=\; 0 \qquad \mathbb{Q}\text{-a.s.},$$

and since our goal is to check (19), this enables us to focus henceforth on

$$\begin{aligned}\tilde{M}(t) \;&=\; M_a(t) - \left(\log\left(1/|\Pi_1(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_1(t)|^{\bar{p}+1} \\ &=\; \sum_{i=2}^{\infty}\left(\log\left(1/|\Pi_i(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_i(t)|^{\bar{p}+1} \qquad \mathbb{Q}\text{-a.s.,}\end{aligned}$$

Next, we compute the conditional expectation of this quantity given $\mathcal{G}_1(\infty)$, the sigma-field generated by the restriction of the discrete point measure to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$, as follows. By construction of the fragmentation Π, each block $\Pi_i(t)$ for $i \geq 2$ got separated from 1 at some instant $r \in \mathcal{D}_1 \cap [0,t]$. More precisely, recall that at such an instant r, the block $\Pi_1(r-)$ splits into $\pi(r)_{|\Pi_1(r-)}$, and that the block after the split which contains 1 is $\Pi_1(r) = \pi_1(r) \cap \Pi_1(r-)$. Thus, there is then some index $j \geq 2$ such that $\Pi_i(t) \subseteq \pi_j(r) \cap \Pi_1(r-)$, where $\pi_j(r)$ stands for the j-th block of the partition $\pi(r)$. In other words, we may consider the partition of $\{2, \ldots\}$ whose blocks are of the type

$$B(r,j) \;=\; \{i \geq 2 : \Pi_i(t) \subseteq \pi_j(r) \cap \Pi_1(r-)\}\;,$$

and then $(\Pi_i(t) : i \in B(r,j))$ forms a partition of $\pi_j(r) \cap \Pi_1(r-)$ which we now analyze.

Lemma 8(ii), standard properties of Poisson random measures, and the very construction of Π entail that for every $r \in [0,t]$ and $j \geq 2$, conditionally on $r \in \mathcal{D}_1$, $\Pi_1(r-)$ and $\pi_j(r)$, the partition $(\Pi_i(t) : i \in B(r,j))$ can be given in the form $\tilde{\Pi}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}$ where $\tilde{\Pi}$ is a homogeneous fragmentation distributed as Π under $\mathbb{P}$ and is independent of the sigma-field $\mathcal{G}_1(\infty)$. Recall from (7) that

$$\mathbb{E}\left(\sum_{i=1}^{\infty} |\Pi_i(t-r)|^{\bar{p}+1}\right) \;=\; \exp(-(t-r)\Phi(\bar{p}))\,,$$

and (taking the derivative)

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \log(1/|\Pi_i(t-r)|)|\Pi_i(t-r)|^{\bar{p}+1}\right) \;=\; (t-r)\Phi'(\bar{p})\exp(-(t-r)\Phi(\bar{p}))\,.$$

The analysis above now entails that

$$\begin{aligned}
&\mathbb{Q}\left(\sum_{i=2}^{\infty} \left(\log\left(1/|\Pi_i(t)|\right) - t\Phi'(\bar{p}) + a\right) \exp\left(t\Phi(\bar{p})\right) |\Pi_i(t)|^{\bar{p}+1} \mid \mathcal{G}_1(\infty)\right) \\
= &\sum_{r\in\mathcal{D}_1\cap[0,t]} \sum_{j=2}^{\infty} \left(\log\left(1/|\pi_j(r)\cap\Pi_1(r-)|\right) - r\Phi'(\bar{p}) + a\right) \exp\left(r\Phi(\bar{p})\right) |\pi_j(r)\cap\Pi_1(r-)|^{\bar{p}+1}\,.
\end{aligned}$$

Now observe that there is the identity

$$|\pi_j(r) \cap \Pi_1(r-)| = |\pi_j(r)||\Pi_1(r-)| = |\pi_j(r)|\exp(-\xi(r-))$$

for all $r \in \mathcal{D}_1$, $\mathbb{P}$-a.s. and hence also $\mathbb{Q}$-a.s. Putting the pieces together, we get

$$\mathbb{Q}(\tilde{M}(t) \mid \mathcal{G}_1(\infty)) \;=\; \sum_{r\in\mathcal{D}_1\cap[0,t]} \exp\left\{-(\bar{p}+1)(\lambda(r-) - a)\right\}\Sigma(r)\,,$$

where

$$\Sigma(r) \;=\; \sum_{j=2}^{\infty} \left(\lambda(r-) - \log|\pi_j(r)|\right)|\pi_j(r)|^{\bar{p}+1}\,.$$

As pointed out by Lyons [22], by the conditional Fatou's theorem, all that we need is to check that $\lim_{t\to\infty}\mathbb{Q}(\tilde{M}(t)\mid\mathcal{G}_1(\infty))<\infty$, $\mathbb{Q}$-a.s. In this direction, we compute the $(\mathbb{Q},\mathcal{G}_1(t))$-predictable compensator corresponding to the point process $\{\Sigma(r), r\in\mathcal{D}_1\}$, and we find

$$\begin{aligned} &\lambda(r-)^{-1}\int_{\mathcal{P}}\mu(d\pi)\,(\lambda(r-)-\log|\pi_1|)\,|\pi_1|^{\bar{p}}\left(\sum_{j=2}^{\infty}(\lambda(r-)-\log|\pi_j|)\,|\pi_j|^{\bar{p}+1}\right)\\ =\;&\lambda(r-)^{-1}\int_{\mathcal{S}}\nu(d\mathbf{s})\left\{\left(\sum_{j=1}^{\infty}(\lambda(r-)-\log|s_j|)\,|s_j|^{\bar{p}+1}\right)^2-\sum_{j=1}^{\infty}(\lambda(r-)-\log|s_j|)^2\,|s_j|^{2\bar{p}}\right\}.\end{aligned}$$

Using the fact that $\bar{p}>0$, it is easily seen that this quantity can be bounded from above by $C\,(\lambda(r-)+1+1/\lambda(r-))$ for some constant C that depends only on ν. So, all that we need now is to verify that the integral

$$\int_0^\infty(\lambda(r)+1+1/\lambda(r))\exp\{-(\bar{p}+1)(\lambda(r)-a)\}\,dr$$

converges $\mathbb{Q}$-a.s., which is immediate from (21). □

Finally, we complete the proof of Proposition 7.

Proof of (iii): To ease the reading, let us denote

$$Y_i(t)=\exp\left(t\Phi(\bar{p})\right)\left(|\Pi(t)|_i^{\downarrow}\right)^{\bar{p}+1}.$$

We first remark that for all $i\in\mathbb{N}$ $\lim_{t\to\infty}Y_i(t)=0$, $\mathbb{P}$-a.s., and we deduce from (18) that $M_a(t)\le -M'(t)+aM(\bar{p},t)$, for t large enough. Taking the limit as $t\to\infty$, we get

$$M_a(\infty)\le -M'(\infty)\,,\qquad \mathbb{P}\text{-a.s.},\tag{22}$$

which proves that $-M'(\infty)\ge 0$ $\mathbb{P}$-a.s.

From the fragmentation property at time 1, we may express $M'(1+t)$ in the form

$$-M'(1+t)\;=\;\sum_{i,j}Y_i(1)Y_{i,j}(t)\log\frac{1}{Y_i(1)Y_{i,j}(t)}\,,$$

where $\{Y_{i,j}(\cdot), j\in\mathbb{N}\}$ for $i=1,\dots$ are independent copies of $\{Y_j(\cdot), j\in\mathbb{N}\}$, which are also independent of $\mathcal{G}(1)$. This yields

$$-M'(1+t)\;=\;\sum_i Y_i(1)\,(-M'_i(t))+\sum_i\left(Y_i(1)\log\frac{1}{Y_i(1)}\right)M_i(t)\tag{23}$$

where $\{M_i(\cdot), i\in\mathbb{N}\}$ (respectively, $\{M'_i(\cdot), i\in\mathbb{N}\}$) are independent copies of $M(\bar{p},\cdot)$ (respectively, of $M'(\cdot)$) and independent of $\mathcal{G}(1)$. To get rid of the last infinite random combination of martingales converging to zero, we first establish the following technical result :

$$\lim_{t\to\infty}\sum_i\left(Y_i(1)\log\frac{1}{Y_i(1)}\right)M_i(t)\;=\;0\qquad\text{in probability under }\mathbb{P}.\tag{24}$$

Indeed, because the $|\Pi(1)|_i^{\downarrow}$, $i \in \mathbb{N}$ are ranked in the decreasing order, and their sum is at most 1, we have $|\Pi(1)|_i^{\downarrow} \leq 1/i$ for every i, and thus $Y_i(1) < 1$ for $i > \mathrm{e}^{\Phi'(\bar{p})}$. The series $-M'(1) = \sum_i Y_i(1)|\log Y_i(1)|$ is absolutely convergent (and in L^1). Therefore, for every $\epsilon > 0$ there exists $k > \mathrm{e}^{\Phi'(\bar{p})}$ such that

$$\mathbb{E}\left(\sum_{k+1}^{\infty} Y_i(1)|\log Y_i(1)|\right) \leq \varepsilon^2 .$$

Since $\mathbb{E}(M_i(t)) = 1$ for all i, the Markov inequality enables us to write

$$\mathbb{P}\left(\sum_{k+1}^{\infty} (Y_i(1)|\log Y_i(1)|)\, M_i(t) > \epsilon\right) \leq \epsilon .$$

Since the sum of the k remaining terms converges $\mathbb{P}$-a.s. to 0, the claim (24) is proved.

Now we are able to complete the proof of (iii). Assume that $\mathbb{P}(M'(\infty) = 0) > 0$. From (23) and (24) we may write

$$M'(\infty) = Y_1(1)M_1'(\infty) + Y_2(1)M_2'(\infty) + B$$

where $B = \lim_t \sum_{i=3}^{\infty}$ is independent of $(M_1'(\infty), M_2'(\infty))$ conditionally on $\mathcal{G}_1$. Since $\mathbb{P}(M'(\infty) \leq 0) = 1$, this entails $\mathbb{P}(B \leq 0) = 1$ and

$$M'(\infty) \leq Y_1(1)M_1'(\infty) + Y_2(1)M_2'(\infty) .$$

This implies $\mathbb{P}(M'(\infty) = 0) \leq \mathbb{P}(M'(\infty) = 0)^2$, so we would have $\mathbb{P}(M'(\infty) = 0) = 1$, which contradicts (ii). □

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