A functional large deviations principle for quadratic forms of Gaussian stationary processes

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Abstract

A functional large deviations principle is proved for quadratic forms of centered stationary Gaussian processes indexed by discrete or continuous time.

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1 Introduction

Let $X = (X_n)_{n\in\mathbb{Z}}$ be a discrete time stationary Gaussian process and $Y = (Y_s)_{s\in\mathbb{R}}$ be a continuous time stationary Gaussian process. Let g_X and g_Y denote the spectral densities of X and Y defined respectively on $\mathbb{T} = [-\pi, \pi]$ and $\mathbb{R}$. They are non negative integrable even functions and the covariance of both processes may be computed as the Fourier transforms of these functions:

$$\begin{aligned} \mathbb{E}(X_n X_{n+k}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt} g_X(t)dt, \quad (n,k\in\mathbb{Z}), && (1)\\ \mathbb{E}(Y_s Y_{s+r}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{irt} g_Y(t)dt, \quad (s,r\in\mathbb{R}). && (2)\end{aligned}$$

In what follows, we will consider the random positive measure built with quadratic forms of X:

$$\nu_n^X(f) = \frac{1}{n}X^{(n)*} f(\Gamma_n) X^{(n)} \quad , f\in\mathcal{C}([m_{g_X}, M_{g_X}]) \tag{3}$$

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where $X^{(n)*} = (X_1, \ldots, X_n)$, Γ_n is the covariance matrix of $X^{(n)}$ and $\mathcal{C}([m_{g_X}, M_{g_X}])$ denotes the set of all continuous functions on $[m_{g_X}, M_{g_X}] = [\text{ess-inf} g_X, \text{ess-sup} g_X]$. In Section 2.1 we recall the meaning of the matrix $f(\Gamma_n)$ (see the proof of Lemma 2). In the continuous case, let Γ_T be the covariance operator of $Y^{(T)} = (Y_s, s \in [0,T])$. It is a trace-class operator from $H_T = L^2([0,T])$ to itself. Its spectrum $\sigma(\Gamma_T)$ lies in $[0, M_{g_Y}]$ where $M_{g_Y} = \text{ess-sup} g_Y$. For any continuous function on $[0, M_{g_Y}]$ we may define the operator $f(\Gamma_T)$ belonging to $L(H_T)$ (see [1]). Let

$$\nu_T^Y(f) := \frac{1}{T}\int_0^T Y_s \, [f(\Gamma_T)Y^{(T)}] \, (s) \, ds = \frac{1}{T}\langle Y^{(T)}, f(\Gamma_T)Y^{(T)}\rangle \tag{4}$$

where $\langle , \rangle$ is the scalar product in $L^2([0,T])$.
The aim of this note is to establish a large deviations principle (L.D.P.) for the random measures ν_n^X and ν_T^Y. For the sake of completeness we recall the definition of a L.D.P. (cf. [7]).

Definition 1 *We say that a sequence (R_n) of probability measures on a measurable Hausdorff space $(U, \mathcal{B}(U))$ satisfies a L.D.P. with rate function I if:*

i) I is lower semi–continuous, with values in $\mathbb{R}^+ \cup \{+\infty\}$.

ii) For any measurable set A of U:

$$-I(intA) \leq \liminf_{n\to\infty} \frac{1}{n} \log R_n(A) \leq \limsup_{n\to\infty} \frac{1}{n} \log R_n(A) \leq -I(cloA),$$

where $I(A) = \inf_{\xi \in A} I(\xi)$.

We say that the rate function I is good if its level sets $\{x \in U : \; I(x) \leq a\}$ are compact for any $a \geq 0$. More generally, a sequence of U-valued random variables is said to satisfy a L.D.P. if their distributions satisfy a L.D.P..

For a fixed f, a L.D.P. for $\nu_n^X(f)$ is already available since it is a particular case of Proposition 3 of Bercu et al [3]. One interest of this paper is to present a functional version of this L.D.P. We give here large deviations principles for random measures with convex but not strictly convex rate functions. It is substantially different from the classical Sanov's case. The main idea here is to adapt the proof of the Baldi's theorem ([7]).
The paper is organized as follows: in Section 2.1 (resp. Section 2.2) we establish a L.D.P. for (ν_n^X) (resp. for (ν_T^Y)). All technical proofs are postponed to Section 2.3.

2 Main Results

2.1 Discrete time case

From now on, we write g, ν_n, m and M for g_X, ν_n^X, m_{g_X} and M_{g_X} respectively.
We assume that the function g is continuous and positive on $\mathbb{T}$, so that $m > 0$. Let $\mathcal{M}([m,M])$ be the set of all positive bounded measures on $[m,M]$. Endowed with the weak topology, it is Polish (metric separable and complete). We first check that (3) defines a random positive measure on $[m,M]$ satisfying a law of large numbers. This is the aim of the following Lemma.

Lemma 2

Let $\lambda_1^n, \dots, \lambda_n^n$ denote the eigenvalues of Γ_n.

a) Almost surely ν_n is in $\mathcal{M}([m,M])$ and there exist independent $\chi^2(1)$–distributed random variables $Z_1^n, \dots, Z_n^n$ such that

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^n Z_i^n \delta_{\lambda_i^n} \tag{5}$$

b) For any f in $\mathcal{C}([m,M])$,

$$\nu_n(f) \rightarrow \nu(f) \quad \text{in probability when} \quad n \rightarrow +\infty\,,$$

where

$$\nu(f) = \int_{[m,M]} f(t)\, t dP(t)\,,$$

and P denotes the image probability of the normalized Lebesgue measure on the torus by the application g, so that

$$\nu(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) f[g(x)] dx\,.$$

Proof:

Let O be an orthogonal $n \times n$ matrix such that $O^*\Gamma_n O$ is the diagonal matrix whose i-th diagonal element is λ_i^n. Let f be in $\mathcal{C}([m,M])$, recall that $f(\Gamma_n)$ is defined by $f(\Gamma_n) = OD_fO^*$ where D_f is the diagonal matrix whose i-th diagonal element is $f(\lambda_i^n)$.
From the Cochran theorem, we may write $\nu_n(f)$ as

$$\nu_n(f) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^n Z_i^n f(\lambda_i^n), \tag{6}$$

where for $n \geq 1$, the random variables $Z_1^n, \dots, Z_n^n$ are independent $\chi^2(1)$–distributed and do not depend on f. Consequently,

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^n Z_i^n \delta_{\lambda_i^n},$$

so that a) of the Lemma 2 follows.
Set P_n the empirical measure

$$P_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^n}. \tag{7}$$

From the theorem of Szegö on Toeplitz forms (see [12] and [2]),

$$P_n \underset{n\rightarrow+\infty}{\Rightarrow} P \tag{8}$$

where $\Rightarrow$ denotes the weak convergence. Moreover, for all $n \geq 1$, the support of P_n is contained in $[m, M]$.

$$n\mathrm{Var}\,(\nu_n(f)) = \frac{2}{n}\sum_{i=1}^{n} \lambda_{i,n}^2 f(\lambda_{i,n})^2 \underset{n\to+\infty}{\to} \frac{1}{\pi}\int_{\mathbf{T}} g(t)^2 f[g(t)]^2 dt\,,$$

which proves b). ■

Now we deal with the large deviations properties of (ν_n). For any f in $\mathcal{C}([m, M])$ set

$$\Lambda(f) = \begin{cases} -\displaystyle\int_{[m,M]} \frac{\log(1-2tf(t))}{2t} d\nu(t) & \text{if} \quad \forall\, t \in [m,\, M]\,,\; tf(t) \leq \dfrac{1}{2} \\ +\infty \text{ otherwise.} \end{cases} \tag{9}$$

and for any μ in $\mathcal{M}([m, M])$ define the convex dual function of Λ by

$$\Lambda^*(\mu) = \sup_{f\in\mathcal{C}([m,M])} \left(\int f(t)\mu(dt) - \Lambda(f)\right). \tag{10}$$

The following Lemma gives another expression for Λ^*. It is a consequence of theorem 5 of Rockafellar [13].

Let for $\tau > 0$,

$$\gamma(\tau) = \frac{1}{2}(\tau - 1 - \log\tau)$$

The function γ is convex (it is the Cramer transform of the $\chi^2(1)$ distribution). Its recession function (see [13]) is $\tau \to \tau/2$.

Lemma 3 *Λ^* is a good convex rate function.*
For any μ in $\mathcal{M}([m, M])$ having, with respect to ν, the Lebesgue decomposition $\mu = l\nu + \eta$, then

$$\Lambda^*(\mu) = \begin{cases} \displaystyle\int_{[m,M]} \frac{\gamma(l(t))}{t} d\nu(t) + \int_{[m,M]} \frac{d\eta(t)}{2t} & \textit{whenever the integrals are defined} \\ +\infty \textit{ otherwise.} \end{cases} \tag{11}$$

When the integrals are defined, Λ^* can also be rewritten as

$$\Lambda^*(\mu) = \frac{1}{2\pi}\int_{]-\pi,\,\pi]} \gamma(l \circ g(t))\,dt + \int_{[m,\,M]} \frac{d\eta(t)}{2t}.$$

The main result of this section follows.

Theorem 4 *(ν_n) satisfies a L.D.P. with good rate function Λ^*.*

Corollary 5 *Almost surely $\nu_n \Rightarrow \nu$.*

Remark: Formula (11) shows that the case $m = 0$ is more delicate. Actually, it may be tackled as in the continuous time case.

Proof: A result analogous to Theorem 4 was established in another framework in [9] with both time-invariant discretization and random weights (see also [8] and [6]).

The proof is based on the ideas of Baldi's theorem (theorem 4.5.20 of [7]). The framework will not exactly be the same, as the limit normalized cumulant will not be defined everywhere but a careful study of the exposed points shows that the proof can be adapted. For any function f in $\mathcal{C}([m,M])$, the normalized cumulant generating function of ν_n is

$$\Lambda_n(f) = \frac{1}{n}\log E\Big(\exp[n\nu_n(f)]\Big) = \frac{1}{n}\log E\left(\prod_{j=1}^{n}\exp(\lambda_j^n f(\lambda_j^n) Z_j^n)\right).$$

There are three cases to consider for the asymptotic behavior of $\Lambda_n(f)$.

Case 1: $\forall t \in [m,M]$, $t\,f(t) < 1/2$,

$$\Lambda_n(f) = -\frac{1}{2n}\sum_{j=1}^{n}\log\Big(1 - 2\lambda_j^n f(\lambda_j^n)\Big),$$

and therefore from (8),

$$\lim_{n\to+\infty}\Lambda_n(f) = -\int_{[m,M]}\frac{\log(1-2tf(t))}{2t}d\nu(t) = \Lambda(f).$$

Case 2: $\exists t \in [m,M]$ such that $t\,f(t) > 1/2$. For n large enough, there exists i in $\{1,\ldots,n\}$ such that $\lambda_i^n f(\lambda_i^n) > 1/2$ and $\Lambda_n(f) = +\infty$. Therefore

$$\lim_{n\to+\infty}\Lambda_n(f) = \Lambda(f) = +\infty.$$

Case 3: $\forall t \in [m,M]$, $tf(t) \le 1/2$ and $\exists t \in [m,M]$ such that $tf(t) = 1/2$. We do not know in general what happens for the asymptotic behavior of $\Lambda_n(f)$. Nevertheless it does not matter: to find a L.D.P. for ν_n, we will show that it is enough to consider functions in cases 1 or 2, since they are dense in the set of exposing hyperplanes.

Upper bound: The upper bound holds for any compact set of $\mathcal{M}([m,M])$ in view of theorem 4.5.3 b) of [7] and the following Lemma, whose proof is postponed in Section 2.3

Lemma 6 *For all $\delta > 0$ and μ in $\mathcal{M}([m,M])$, there exists f_δ in $\mathcal{C}([m,M])$ such that $tf_\delta(t) < 1/2$ for all t and*

$$\int f_\delta(t)\mu(dt) - \Lambda(f_\delta) \ge \min\{\Lambda^*(\mu) - \delta, \frac{1}{\delta}\}. \tag{12}$$

Exponential tightness: For all $a > 0$,

$$\left\{\sup_{\|f\|_\infty \le 1}\nu_n(f) \ge a\right\} \subset \{\nu_n(1) \ge a\}.$$

Fix θ in $]0, 1/4M[$, the classical Chernov bound and (8) give:

$$\limsup_n \frac{1}{n}\log \mathbb{P}(\nu_n(1) > a) \le -\theta a - \frac{1}{2}\int_{[m,M]}\log(1-2\theta x)\,dP(x),$$

and

$$\lim_{a\to+\infty}\limsup_n \frac{1}{n}\log \mathbb{P}(\nu_n(1)>a)=-\infty\,.$$

Hence the sequence (ν_n) is exponentially tight, and the upper bound holds for any closed set of $\mathcal{M}([m,M])$.

Lower bound: We study the set of the exposed points of Λ^*. We recall that μ in $\mathcal{M}([m,M])$ is an exposed point of Λ^* if there exists a function f of $\mathcal{C}([m,M])$ such that for all ζ in $\mathcal{M}([m,M])$ with $\zeta\neq\mu$,

$$\int_{[m,M]} f(t)\mu(dt)-\Lambda^*(\mu) > \int_{[m,M]} f(t)\zeta(dt)-\Lambda^*(\zeta). \tag{13}$$

The function f is called the exposing hyperplane.

Denote by $\mathcal{F}$ the set of the absolutely continuous measures with respect to ν having a positive continuous Radon-Nikodym derivative. Then $\mathcal{F}$ is a subset of $\mathcal{M}([m,M])$, and the following Lemma shows that $\mathcal{F}$ is a set of exposed points of Λ^*:

Lemma 7 *Let $\mu=l\nu$ be in $\mathcal{F}$. Set*

$$f(t)=\frac{1}{2t}(1-\frac{1}{l(t)}),\quad t\in[m,M]\,.$$

Then μ is an exposed point of Λ^* with exposing hyperplane f.
Furthermore, for some $\gamma>1$, we have $\Lambda(\gamma f)<+\infty$.

(The proof is given in Section 2.3).

Now, from the following Lemma, $\mathcal{F}$ is a dense subset of $\mathcal{M}([m,M])$ and this implies the lower bound .

Lemma 8 *Let μ be in $\mathcal{M}([m,M])$ such that $\Lambda^*(\mu)<+\infty$. There exists a sequence of positive functions (l_n) in $\mathcal{C}([m,M])$ such that $l_n\nu\Rightarrow\mu$ and $\lim_{n\to+\infty}\Lambda^*(l_n\nu)=\Lambda^*(\mu)$.*

(The proof is given in Section 2.3)

2.2 Continuous time case

From now on, we write g, M and ν_T for g_Y, M_{g_Y} and ν_T^Y respectively.

We assume that g is continuous on $\mathbb{R}$, and we follow the scheme of the discrete time case. Let $\mathcal{M}([0,M])$ be the set of the positive bounded measures on $[0,M]$ endowed with the weak topology, and let $\mathcal{C}([0,M])$ be the set of the continuous functions on $[0,M]$. The eigenvalues $\{\lambda_k^{(T)}\}_{k\geq1}$ of Γ_T satisfy $0<\lambda_k^{(T)}\leq M$ and $\sum_{k\geq1}\lambda_k^{(T)}<+\infty$.

Lemma 9

a) Almost surely ν_T defined by (4) is a positive bounded measure and there exists a sequence $\{Z_k^{(T)}\}_{k\geq1}$ of independent $\chi^2(1)$ distributed random variables such that

$$\nu_T=\frac{1}{T}\sum_{k=1}^{\infty}Z_k^{(T)}\lambda_k^{(T)}\delta_{\lambda_k^{(T)}}.$$

b) For any f in $\mathcal{C}([0,M])$,

$$\nu_T(f) \to \nu(f) \quad \textit{in probability as} \quad T \to +\infty\,,$$

where

$$\nu(f) = \int_{-\infty}^{+\infty} g(y) f(g(y)) \frac{dy}{2\pi}.$$

Proof : Let $(e_k^{(T)})$ be a complete orthonormal system of eigenvectors of Γ_T with associated eigenvalues $\{\lambda_k^{(T)}\}$, then

$$\begin{aligned} Y^{(T)} &= \sum_{k\geq 1} \langle Y^{(T)}, e_k^{(T)} \rangle e_k^{(T)} \\ f(\Gamma_T) Y^{(T)} &= \sum_{k\geq 1} f(\lambda_k^{(T)}) \langle Y^{(T)}, e_k^{(T)} \rangle e_k^{(T)} \\ \nu_T^Y(f) &= \frac{1}{T} \sum_{k\geq 1} \lambda_k^{(T)} f(\lambda_k^{(T)}) Z_k^T \end{aligned}$$

where $Z_k^T = (\lambda_k^{(T)})^{-1} \langle Y^{(T)}, e_k^{(T)} \rangle^2$ are independent and $\chi^2(1)$ distributed. So we have,

$$\nu_T^Y = \frac{1}{T} \sum_{k=1}^{\infty} Z_k^T \lambda_k^{(T)} \delta_{\lambda_k^{(T)}}.$$

This defines a positive finite measure since $\sum_{k=1}^{\infty} \lambda_k^{(T)}$ is finite. From Ginovian [11]

$$Q_T = \frac{1}{T} \sum_{k=1}^{\infty} \lambda_k^{(T)} \delta_{\lambda_k^{(T)}} \underset{T\to+\infty}{\Rightarrow} \nu \tag{14}$$

(see also Bryc-Dembo [5] and the seminal book of Grenander-Szegö [12], p.139). To show the convergence of $\nu_T(f)$, we notice that

$$T E[\nu_T(f) - E\nu_T(f)]^2 \to \frac{1}{\pi} \int g(x)^2 f(g(x))^2 \, dx\,. \ \blacksquare$$

By analogy with the discrete–time case, for any function f in $\mathcal{C}([0,M])$ set

$$\Lambda(f) = \begin{cases} -\displaystyle\int_{[0,M]} \frac{\log\left(1 - 2tf(t)\right)}{2t} d\nu(t) & \text{if} \quad \forall\, t \in [0,\, M]\,,\ tf(t) \leq \dfrac{1}{2} \\ +\infty \text{ otherwise.} \end{cases} \tag{15}$$

For any μ in $\mathcal{M}([0,M])$, define the Fenchel-Legendre dual of Λ:

$$\Lambda^*(\mu) = \sup_{f\in\mathcal{C}([0,M])} \left(\int f(t)\mu(dt) - \Lambda(f) \right). \tag{16}$$

In view of Rockafellar [13], we have the following Lemma:

Lemma 10 *Λ^* is a good convex rate function.*
Let μ in $\mathcal{M}([0,M])$ having, with respect to ν, the Lebesgue decomposition $\mu = l\nu + \eta$. If $t \to \gamma(l(t))/t$ is in $L^1(\nu)$ and $1/t$ is in $L^1(\eta)$ then

$$\begin{aligned}\Lambda^*(\mu) &= \int_{[0,M]} \frac{\gamma(l(t))}{t}\, d\nu(t) + \int_{[0,M]} \frac{d\eta(t)}{2t} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(l \circ g(y))dy + \int_{[0,M]} \frac{d\eta(t)}{2t}\,.\end{aligned}$$

Otherwise, $\Lambda^*(\mu) = +\infty$.

Theorem 11 *(ν_T) satisfies a L.D.P. with good rate function Λ^*.*

The proof of this Theorem is similar to the one of Theorem 4. Remark that for all f in $\mathcal{C}([0,M])$ such that $tf(t) < 1/2$, the normalized cumulant generating function of ν_T on f is

$$\Lambda_T(f) = -\int_{[0,M]} \frac{\log(1-2tf(t))}{2t} dQ_T(t)$$

which, from (14) converges towards $\Lambda(f)$ as T goes to $+\infty$.
The analogues of Lemmas 7 and 8 are a little different here, since there is a problem of integrability in 0.

Denote by $\mathcal{G}$ the subset of $\mathcal{M}([0,M])$ of measures $\mu = l\nu$ such that l is in $\mathcal{C}([0,M])$ and satisfies the condition

$$(\mathcal{A}) \qquad \begin{array}{ll} \lim_{t\to 0} l(t) = 1; & \lim_{t\to 0}(1-l(t))/t \text{ is finite.} \\ \forall t \in [0,M]\,, & l(t) > 0\,. \end{array}$$

Lemma 12

a) Let $\mu = l\nu$ be in $\mathcal{G}$. Set

$$f(t) = \frac{1}{2t}\left(1 - \frac{1}{l(t)}\right), \quad t \in [0,M]\,.$$

Then μ is an exposed point of Λ^* with exposing hyperplane f.

b) Let μ be in $\mathcal{M}([0,M])$ such that $\Lambda^*(\mu) < +\infty$. There exists (l_n) in $\mathcal{C}([0,M])$ such that $l_n\nu \Rightarrow \mu$, $\lim_{n\to+\infty} \Lambda^*(l_n\nu) = \Lambda^*(\mu)$ and for all n, l_n satisfies $(\mathcal{A})$.

To complete the proof of Theorem 11, it remains to show the exponential tightness. It is a classical Chernoff bound; the difference with the discrete time case is that we integrate on the tapered measure Q_T. ■

2.3 Proofs of Lemmas 6, 7, 8, 12

2.3.1 Proof of Lemma 6

From the definition of Λ^*, for all $\delta > 0$, there exists f in $\mathcal{C}([m, M])$ such that (12) holds. In case $tf(t) \leq 1/2$, we may add some $\varepsilon < 0$ so that (12) holds with another δ.

2.3.2 Proof of Lemma 7

The proof is based on the strict convexity of the function γ: for two non negative real numbers x, y such that $x \neq y$

$$\gamma(x) - \gamma(y) < (x - y)\gamma'(y) \tag{17}$$

and

$$\gamma'(x) = \frac{1}{2}(1 - \frac{1}{x}).$$

Integrating the relation (17), we obtain (13).
In addition, for any μ in $\mathcal{F}$ the associated exposing hyperplane f is strictly bounded by $1/(2t)$. Consequently, for some $\gamma > 1$, we have $\Lambda(\gamma f) < +\infty$.

2.3.3 Proof of Lemma 8

This Lemma has been proved in the unpublished paper [10]. For sake of completeness, we recall this proof.

Easy considerations on the properties of this function lead to

$$\gamma(\tau + \tau') \leq \gamma(\tau) + \frac{\tau'}{2} \quad (\tau > 0, \tau' \geq 0). \tag{18}$$

Let l and $\tilde{l}$ be non negative measurable functions on $[m, M]$. Integrating (18),

$$\Lambda^*((l + \tilde{l})\nu) \leq \Lambda^*(l\nu) + \frac{1}{2}\int \tilde{l}(t)d\nu \tag{19}$$

whenever the terms on the right are defined.

First step

If $\mu = l\nu + \eta$, with $\Lambda^*(\mu) < \infty$, l non negative and continuous, and the singular part η such that $1/t$ in $L^1(\eta)$. Since P has full support on $[m, M]$, there exists a sequence (h_n) of positive functions in $\mathcal{C}([m, M])$ such that $h_n\, d\nu \Rightarrow \eta/t$. From the lower semi-continuity of Λ^*,

$$\liminf_{n \to +\infty} \Lambda^*((l + th_n)\nu) \geq \Lambda^*(\mu).$$

From inequality (18),

$$\Lambda^*((l + th_n)\nu) \leq \Lambda^*(l\nu) + \frac{1}{2}\int_{[m,M]} h_n\, d\nu$$

And then

$$\liminf_{n\to+\infty} \Lambda^*((l+th_n)\nu) \leq \Lambda^*(\mu)\,.$$

We now show that the Lemma is true if $\mu = l\,\nu$ with l ν-a.s. non negative integrable.

Second step

We prove the result for $\mu = l\,\nu$ assuming that l is integrable and that for some $\epsilon > 0$, $l > \epsilon$ ν-a.s. There exists a sequence (l_n) of continuous positive functions such that l_n converges both in $L^1(\nu)$ norm and ν-a.s. to l and $l > \epsilon/2$. Since on $]\epsilon/2, +\infty[$ the function γ is Lipschitzian we claim that the lemma holds.

Third step

Define $l_\epsilon := l\mathbb{I}_{l>\epsilon} + \epsilon\mathbb{I}_{l\leq\epsilon}$. Apply second step and inequality (19) noticing that l_ϵ converges in $L^1(\nu)$ to l and that $l_\epsilon \geq l$.

Last step

For $\mu = l\nu + \eta$, combine first and third step.

2.3.4 Proof of Lemma 12

a) For l in $\mathcal{C}([0,M])$ satisfying $(\mathcal{A})$, $\Lambda^*(l\,d\nu) < \infty$ because $t \to \gamma(l(t))/t$ is continuous, and $f(t) = (1 - 1/l(t))/2t$ is in $\mathcal{C}([0,M])$.

b) We prove the part b) in two steps:
First step: this step is the same as the one of Lemma 8. Remark that $l + th_n$ satisfies condition $(\mathcal{A})$.

Second step: take $\mu = l\nu$ with l and $\gamma(l)/t$ in $L^1(\nu)$.
Fix $\alpha > 0$. We approximate l by l_4 satisfying $(\mathcal{A})$ with the following construction:

1) Let l_1 be the function defined by $l_1(x) = 1$ if $x \in [0,\beta]$, $l(x)$ if $x \in [\beta, M]$, where β is chosen so that $|\Lambda^*(l_1) - \Lambda^*(l)| \leq \alpha$ and $||l_1 - l||_1 \leq \alpha$ (the norm $||.||_1$ is related to the measure ν).

2) Set $l_2 = l_1\mathbb{I}_{l_1>\epsilon} + \epsilon\mathbb{I}_{l_1\leq\epsilon}$, where ϵ is chosen such that $|\Lambda^*(l_2) - \Lambda^*(l_1)| \leq \alpha$ and $||l_2 - l_1||_1 \leq \alpha$. Therefore $l_2 \geq \epsilon$.

3) Set $l_3 = 1$ on $[0,\beta]$ and l_3 is positive continuous on $[\beta, M]$, with $l_3 \geq \epsilon/2$ and $||l_3 - l_2||_1 \leq \alpha$.

4) Let l_4 be in $\mathcal{C}([0,M])$, defined by

$$l_4(x) = \begin{cases} 1 \text{ if } x \in [0,\beta] \\ \text{is linear on } [\beta,\delta] \\ l_3(x) \text{ if } x \in [\delta, M] \end{cases}$$

with δ chosen such that $|\Lambda^*(l_4) - \Lambda^*(l_3)| \leq \alpha$ and $||l_4 - l_3||_1 \leq \alpha$.

Therefore $|\Lambda^*(l_4) - \Lambda^*(l)| \leq 4\alpha$ and $||l_4 - l||_1 \leq 4\alpha$.

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