A LDP related to the Strong Arc-Sine Law

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1 Introduction

For $B = (B_u, u \geq 0)$ a standard linear Brownian motion, a classical result due to Lévy (1939 [29]) asserts that the distribution of $\frac{1}{T}\int_0^T 1_{B_u>0}\, du$ is the arc-sine law. Taking a logarithmic weight yields the so-called Lévy's strong arc-sine law ([29]). It says that

$$S_T := \frac{1}{\log T}\int_1^T 1_{(B_u>0)}\frac{du}{u} \to \frac{1}{2} \quad (T\to\infty) \text{ a.s. .} \tag{1.1}$$

For a simple proof, let us introduce for every $s > 0$ the rescaled process

$$\sigma_s B(u) = s^{-1/2}B_{su}, \quad u \geq 0.$$

These maps define a continuous action of the multiplicative group $\mathbb{R}^+$ on $\mathcal{C}([0,\infty),\mathbb{R})$. The Wiener measure is invariant by the ergodic shift σ_s, so Birkhoff's theorem yields the above convergence.

Actually this can be seen in another way using the stationary Ornstein-Uhlenbeck process

$$X_u = e^{-u/2}B(e^u) \quad -\infty < u < +\infty. \tag{1.2}$$

Denoting by $A_t^+ = \int_0^t 1_{(X_s>0)}\, ds$ the time spent by the process X above 0 before t, the standard ergodic theorem gives

$$\lim_{t\to\infty}\frac{A_t^+}{t} = \frac{1}{2} \ a.s.,$$

so setting $t = \log T$ yields (1.1). For a bibliography on this subject, we refer to Bingham [1], and to Bertoin–Werner [4] for some similar use of the BM-OU correspondence.

In this paper, we prove the following theorem.

Theorem 1.1

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{\log T}\ \log P(S_T \geq a) &= -\mathbf{J}(a) \quad \textit{for } 1/2 \leq a \leq 1 \\ \lim_{T\to\infty}\frac{1}{\log T}\ \log P(S_T \leq a) &= -\mathbf{J}(a) \quad \textit{for } 0 \leq a \leq 1/2. \end{aligned}$$

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where $\mathbf{J}(x) = \sup_\lambda\big(\lambda x - \Lambda(\lambda)\big)$ *and* $\Lambda(\lambda)$ *is the unique solution* $\alpha \in (-1/2, +\infty) \cap (-1/2 + \lambda, +\infty)$ *of the equation*

$$\frac{\Gamma(\alpha - \lambda + \frac{1}{2})}{\Gamma(\alpha - \lambda)} + \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)} = 0. \tag{1.3}$$

In other words, the family of distributions of $(S_T, T > 1)$ satisfies a Large Deviation Principle (LDP) at speed $(\log T)^{-1}$ with the convex good rate function $\mathbf{J}$ (see Dembo-Zeitouni [16] for the definition). Actually we prove the equivalent statement on the family of distributions of $(\frac{A_t^+}{t}, t > 0)$ at speed t^{-1}. We use the method of the Laplace transform (so-called Gärtner-Ellis theorem [16]), i.e. we study $\Lambda(-\lambda) = \lim_{t\to\infty} \frac{1}{t} \log Ee^{-\lambda A_t^+}$. A key role is played by the function $b(\alpha) = \frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha}{2})}$ which is the Laplace exponent of the inverse local time in zero.

Since $A_t^+ = L_t(0, \infty)$ where $L_t := \frac{1}{t}\int_0^t \delta_{X_u} du$ is the empirical distribution, we may also use known results on large deviations for L_t and use the contraction principle.

For the empirical distribution, LDP is due to Donsker-Varadhan [14] (for the weak topology of measures): the rate function is given by

$$\mathbf{I}(\mu) := \frac{1}{2}\int (g')^2 d\mathcal{N}$$

if $\mathcal{N}$ is the standard normal distribution and $d\mu = g^2 d\mathcal{N}$.

At the process level, the LDP for OU is known ([15], [9], [11]): the rate function is the specific entropy $\mathbf{H}$. Actually the LDP holds for the τ-topology, i.e. induced by the duality with bounded measurable functions ([11], [8]).

By the contraction principle, we see that

$$\mathbf{J}(a) = \inf\{\mathbf{I}(\mu);\ \mu([0, +\infty]) = a\}\,. \tag{1.4}$$

The paper is organized as follows. In section 2 we use the excursion theory for the OU process to compute Λ and J. In section 3, we consider the variational problem issued from the contraction principle and exhibit a measure μ_a such that $\mathbf{I}(\mu_a) = \mathbf{J}(a)$. It is the invariant measure of an ergodic diffusion whose distribution Q_a satisfies $\mathbf{H}(Q_a) = \mathbf{I}(\mu_a) = \mathbf{J}(a)$ (and it is the only one). So we can say that to reach the mean occupation time proportion $a \neq 1/2$, the cost is $\mathbf{J}(a)$ and the way is to follow Q_a. In section 4, we give an extension of computations of section 1 to Walsh processes. In section 5, we prove a LDP for the partial sum version of the strong arcsine law. It is connected to the LDP for the Almost Sure Central Limit Theorem ([21], [22], [30])

2 Proof of Theorem 1

We will deduce the rate function from a Laplace transform computation. A direct expression of the Laplace transform of $\int_0^t 1_{(X_s>0)} ds$ seems difficult to obtain, but we get a closed expression as we replace t by an exponential variable of parameter α, independent of the stationary. First we study a OU process starting from 0, which allows to use excursion theory. This yields the Laplace transform in this case. The Markov property and integration with respect to the invariant measure give the Laplace transform in the stationary case.

2.1 Laplace transforms

Let us denote by $(U_t^{(c)},\ t \geq 0)$ the OU process of dimension 1 and parameter c, i.e. the solution of

$$dU_t = dB_t - cU_t dt, \quad U_0 = 0, \tag{2.1}$$

when $(B_t, t \geq 0)$ is a one-dimensional Brownian motion; we write $P^{(c)}$ for the law of $U^{(c)}$ on $\mathcal{C}(\mathrm{I\!R}_+, \mathrm{I\!R})$. We are interested in $E^{(c)}[e^{-\lambda A_t^+}]$ where $A_t^+ = \int_0^t 1_{(U_s>0)} ds$. Our main tool will be the double Laplace transform

$$\mathcal{L}^{(c)}(\lambda, \alpha) := \int_0^\infty e^{-\alpha t} E^{(c)} \left[e^{-\lambda A_t^+}\right] dt\,,$$

which is defined on $\mathbb{R}^2$, valued in $[0, +\infty]$. Moreover, if A_t^- is the time spent below 0 before t, then $A_t^+ + A_t^- = t$, and by symmetry A_t^+ and A_t^- have the same law, so that

$$\mathcal{L}^{(c)}(\lambda, \alpha) = \mathcal{L}^{(c)}(-\lambda, \alpha + \lambda)$$

which entails that $\mathcal{L}^{(c)}$ is a symmetric function of α and $\alpha + \lambda$. Since $0 \leq A^+(t) \leq t$, the quantity $E^{(c)} e^{-\lambda A^+(t)}$ is finite for every $\lambda \in \mathrm{I\!R}$, hence $\mathcal{L}^{(c)}$ is clearly finite for $\alpha > 0$, $\alpha + \lambda > 0$. The computations of this paragraph will be done in that case.

From the scaling property of $(B_t, t \geq 0)$, one obtains that for any $\kappa > 0$,

$$\left(\frac{1}{\kappa} U_{t\kappa^2}^{(c)},\ t \geq 0\right) \stackrel{\mathcal{D}}{=} \left(U_t^{(c\kappa^2)},\ t \geq 0\right) \tag{2.2}$$

and consequently:

$$\mathcal{L}^{(c)}(\lambda, \alpha) = \frac{1}{c} \mathcal{L}^{(1)}\left(\frac{\lambda}{c}, \frac{\alpha}{c}\right) \tag{2.3}$$

Thus, it suffices to compute,

$$\mathcal{L}^{(1)}(\lambda, \alpha) =: \mathcal{L}(\lambda, \alpha).$$

For a general one-dimensional diffusion $(X_t, t \geq 0)$, for which 0 is regular for itself and a Borel function $q : \mathbb{R} \to \mathbb{R}^+$, the excursion theory for X (see [31], [32]) allows to express:

$$\mathcal{L}(q, \lambda, \alpha) := E\Big[\int_0^\infty dt\ e^{-\alpha t} e^{-\lambda \int_0^t q(X_s) ds}\Big],$$

as:

$$\mathcal{L}(q, \lambda, \alpha) = \frac{N_q(\lambda, \alpha)}{D_q(\lambda, \alpha)} \tag{2.4}$$

where

$$N_q(\lambda, \alpha) = \int n_X(d\epsilon) \int_0^{V(\epsilon)} du\ e^{-\int_0^u dh\ \tilde{q}(\epsilon_h)}$$

$$D_q(\lambda, \alpha) = \int n_X(d\epsilon) \Big(1 - e^{-\int_0^{V(\epsilon)} dh \tilde{q}(\epsilon_h)}\Big)$$

with $\tilde{q}(x) = \alpha + \lambda q(x)$, and $n_X(d\epsilon)$ is Itô's excursion measure associated with X, together with a choice of its local time $(l_u, u \geq 0)$ at 0. Here we choose the local time as density of the occupation time measure with respect to the Lebesgue measure

$$l_t = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t 1_{(-\delta,+\delta)}(X_s)ds \tag{2.5}$$

and the inverse local time as

$$\tau_t = \inf\{u : l_u > t\}.$$

In our particular case $q(x) = 1_{(x>0)}$. Moreover, Itô's measure is symmetric since $U \overset{\mathcal{D}}{=} -U$. Formula (2.4) becomes

$$\mathcal{L}(\lambda, \alpha) = \frac{a(\alpha+\lambda) + a(\alpha)}{b(\alpha+\lambda) + b(\alpha)}, \tag{2.6}$$

with, moreover $a(\mu) = \frac{b(\mu)}{\mu}$, and $b(\mu)$ is the Lévy exponent featured in the formula:

$$E\Big[\exp\Big(-\mu \int_0^{\tau_t} ds 1_{(U_s>0)}\Big)\Big] = \exp\Big(-tb(\mu)\Big). \tag{2.7}$$

Actually

$$b(\mu) = \int n(d\epsilon)(1 - e^{-\mu V^+(\epsilon)}) = \int_0^\infty n_V^+(ds)(1 - e^{-\mu s}) \tag{2.8}$$

where V^+ is the duration of a positive excursion and n_V^+ is the Lévy measure of $A^+_{\tau_t}$.

Analogous results may be found in [3] and in [39].

Again owing to the symmetry of U and the consequence of excursion theory that the processes $\Big(\int_0^{\tau_t} ds 1_{(U_s>0)},\ t \geq 0\Big)$ and $\Big(\int_0^{\tau_t} ds 1_{(U_s<0)},\ t \geq 0\Big)$ are independent, we get

$$E\Big[\exp(-\mu\tau_t)\Big] = \exp\Big(-2tb(\mu)\Big). \tag{2.9}$$

Formulas for b may be found in [20] (formula 22, p.96) and in [5] (formula 7.4.0.1 p. 442),

$$b(\mu) = \frac{\Gamma\Big(\frac{1}{2} + \frac{\mu}{2}\Big)}{\Gamma\Big(\frac{\mu}{2}\Big)} \tag{2.10}$$

Formulas for n_V^+ may be found in [20] and in [31], formula (59) (with another normalisation of the local time); here

$$n_V^+(dv) = \frac{1}{\Gamma(1/2)} e^{-v}(1 - e^{-2v})^{-3/2} dv. \tag{2.11}$$

Gathering (2.3), (2.6) and (2.10) yields

$$\mathcal{L}^{(c)}(\lambda, \alpha) = \frac{1}{c} \frac{\Bigg(\frac{b\Big(\frac{\lambda+\alpha}{c}\Big)}{\Big(\frac{\lambda+\alpha}{c}\Big)} + \frac{b\Big(\frac{\alpha}{c}\Big)}{\Big(\frac{\alpha}{c}\Big)}\Bigg)}{\Big(b\Big(\frac{\lambda+\alpha}{c}\Big) + b\Big(\frac{\alpha}{c}\Big)\Big)} \tag{2.12}$$

It remains to apply formula (2.10). In the case $c = 1/2$, formula (2.12) simplifies as follows:

$$\mathcal{L}^{(1/2)}(\lambda,\alpha) = \frac{\dfrac{\Gamma(\alpha+\lambda+\frac{1}{2})}{(\alpha+\lambda)\Gamma(\alpha+\lambda)} + \dfrac{\Gamma(\alpha+\frac{1}{2})}{\alpha\Gamma(\alpha)}}{\dfrac{\Gamma(\alpha+\lambda+\frac{1}{2})}{\Gamma(\alpha+\lambda)} + \dfrac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}} \tag{2.13}$$

For $x \in \mathbb{R}$, let P_x be the law of the OU process of dimension 1 and parameter 1/2 with initial distribution δ_x and

$$\mathcal{L}_x(\lambda,\alpha) := \int_0^\infty e^{-\alpha t} E_x\left[e^{-\lambda A_t^+}\right] dt,$$

in particular $\mathcal{L}_0(\lambda,\alpha) = \mathcal{L}^{1/2}(\lambda,\alpha)$. Let P^* be the law of the OU process under its stationary distribution (in this case $\mathcal{N}(0,1)$), and

$$\mathcal{L}^*(\lambda,\alpha) = \int_0^\infty e^{-\alpha t} E^*\left[e^{-\lambda A_t^+}\right] dt = \int_{\mathbb{R}} \mathcal{L}_x(\lambda,\alpha) d\mathcal{N}(x)\,. \tag{2.14}$$

Let us first give an expression of $\mathcal{L}_x$, and then $\mathcal{L}^*$.

We may write

$$\alpha\mathcal{L}_x(\lambda,\alpha) = E_x e^{-\lambda A^+_{\gamma_\alpha}} \tag{2.15}$$

where γ_α is an independent exponential time, with parameter α, that is :$P(\gamma_\alpha \in dt) = \alpha e^{-\alpha t} dt$. Borodin and Salminen [5] Formula II 7.1.4.1 give an expression for the RHS of (2.15). In order to get later $\mathcal{L}^*$, let us give some details.

Let us assume for instance $x > 0$ and let T_0 be the first hitting time of 0. If $T_0 > t$, then $A_t^+ = t$. Otherwise, we have to decompose $[0,t] = [0,T_0[\cup[T_0,t]$ and $A_t^+ = T_0 + A^+(t-T_0)\circ\theta_{T_0}$ where θ_s is the shift at time s. Now

$$\int_0^\infty e^{-\alpha t} E_x\left[e^{-\lambda A_t^+}\right] dt = E_x \int_0^{T_0} e^{-\alpha t - \lambda t} dt + \int_{T_0}^\infty e^{-\alpha t - \lambda T_o - \lambda A^+(t-T_0) o \theta_{T_0}} dt\,. \tag{2.16}$$

Defining $f(x,\alpha) := E_x e^{-\alpha T_0}$, and applying the strong Markov property, we see that the second integral in the RHS of (2.16) is $f(x,\alpha+\lambda)\mathcal{L}_0(\lambda,\alpha)$. This allows to write

$$\mathcal{L}_x(\lambda,\alpha) = \frac{1}{\alpha+\lambda}[1 - f(x,\alpha+\lambda)] + f(x,\alpha+\lambda)\mathcal{L}_0(\lambda,\alpha) \tag{2.17}$$

A similar study for $x < 0$ gives

$$\mathcal{L}_x(\lambda,\alpha) = \frac{1}{\alpha}[1 - f(x,\alpha)] + f(x,\alpha)\mathcal{L}_0(\lambda,\alpha). \tag{2.18}$$

Integrating (2.17) and (2.18) yields

$$\mathcal{L}^*(\lambda,\alpha) = \frac{1}{2}\Big[\frac{1}{\alpha+\lambda} + \frac{1}{\alpha}\Big] - \frac{f^*(\alpha+\lambda)}{\alpha+\lambda} - \frac{f^*(\alpha)}{\alpha} + \mathcal{L}_0(\lambda,\alpha)[f^*(\alpha+\lambda) + f^*(\alpha)] \tag{2.19}$$

where we set

$$f^*(\alpha) := E^*[e^{-\alpha T_0} 1_{(X_0>0)}] = \int_0^\infty f(x,\alpha) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \tag{2.20}$$

Proposition 2.1

$$f^*(\alpha) = \frac{b(2\alpha)}{2\alpha\sqrt{\pi}}\,. \tag{2.21}$$

Proof:

a) This result is the content of Lemma p.129 in [37] or Proposition 1 of [38] where it is proved for a one-dimensional time-homogeneous diffusion with infinitesimal generator $\frac{1}{2}\frac{d^2}{dx^2} + \rho(x)\frac{d}{dx}$. The proof uses the Schrödinger operator associated with the diffusion and an expression of $f(x,\alpha)$ in terms of the Green function.

b) Another interesting frame for this result is the general "local" formula (63)-(70) given in [31] for diffusions whose generator is $\frac{1}{2}\frac{d}{dm}\frac{d}{dx}$. Actually all the above computations may have been performed for strings (see [25] for the definition and [39] for related results).

Let us give another probabilistic proof without use of the spectral theory (see [31] example 7 and its relation with a "global" formula). From (1.2) we have

$$T_0 = \inf\{t \geq 0 : X_t = 0\} = \inf\{t \geq 0 : B(e^t) = 0\}$$

hence

$$\begin{aligned} e^{T_0} &= \inf\{u \geq 1 : B(u) = 0\} \\ &= 1 + \inf\{v \geq 0 : B(v+1) - B(1) = -B(1)\}. \end{aligned}$$

Since $(B(v+1) - B(1), v \geq 0)$ is a BM independent of $B(1)$ we have

$$e^{T_0} \stackrel{\mathcal{D}}{=} 1 + \frac{B(1)^2}{\tilde{B}(1)^2}$$

where $\tilde{B}(1)$ is $\mathcal{N}(0,1)$ independent of $B(1)$. This entails

$$e^{T_0} \stackrel{\mathcal{D}}{=} \frac{1}{A_0} \tag{2.22}$$

where A_0 has the arc-sine distribution, and, by independence of $|\beta_1/B_1|$ and $\mathrm{sign}(\beta_1)$,

$$f^*(\alpha) = \frac{1}{2}E\big[A_0^\alpha\big] = \frac{1}{2\pi}B\big(\frac{2\alpha+1}{2}, \frac{1}{2}\big)$$

where $B(a,b)$ is the beta function. This coincides with (2.21) in view of (2.10).

Actually, for the same reasons,

$$E^*\big[h(T_0); X_0 > 0\big] = \frac{1}{2}E\big[h(-\log A_0)\big]. \qquad \square \tag{2.23}$$

The complete formula giving $\mathcal{L}^*(\lambda,\alpha)$ is obtained gathering (2.13),(2.19) and (2.21). We get

$$\mathcal{L}^*(\lambda,\alpha) = \Phi(\alpha, \alpha+\lambda)\,, \tag{2.24}$$

where Φ is the symmetrical function

$$\Phi(\alpha,\beta) = \frac{1}{2\alpha}\left(1 - \frac{b(2\alpha)}{\alpha\sqrt{\pi}}\right) + \frac{1}{2\beta}\left(1 - \frac{b(2\beta)}{\beta\sqrt{\pi}}\right) + \frac{1}{2\sqrt{\pi}}\frac{\left[\dfrac{b(2\alpha)}{\alpha} + \dfrac{b(2\beta)}{\beta}\right]^2}{b(2\alpha)+b(2\beta)}. \tag{2.25}$$

Remark 2.2 *We might also have found (2.13) using formula (1) p.117 of ([37]), which reads:*

$$\mathcal{L}^{1/2}(\lambda,\alpha) = \frac{(\alpha+\lambda)^{-1} f'_x(0^+,\alpha+\lambda) - \alpha^{-1} f'_x(0^-,\alpha)}{f'_x(0^+,\alpha+\lambda) - f'_x(0^-,\alpha)}. \tag{2.26}$$

2.2 Domains

To get a LDP, we need to know the whole domain of finiteness of $\mathcal{L}^*$. This is the object of the present paragraph. More precisely, for any $\lambda \in \mathbb{R}$, we look for

$$\mathcal{D}_\lambda := \{\alpha \in \mathbb{R} : \mathcal{L}^*(\lambda,\alpha) < \infty\}. \tag{2.27}$$

As a Laplace transform, $\mathcal{L}^*(\lambda,.)$ is lower semicontinuous and convex, so that $\overset{\circ}{\mathcal{D}}_\lambda$ is an interval $(\alpha(\lambda),+\infty)$. We already know that $\mathcal{L}^*(\lambda,.)$ coincides with the right hand side (RHS) of (2.19) on $\alpha \in (0,+\infty) \cap (-\lambda,+\infty)$. Since $\mathcal{L}^*(\lambda,.)$ is analytic, $\mathcal{D}_\lambda$ is the largest convex set on which this RHS can be extended by analytic continuation.

Formula (2.8) allows to define a semicontinuous concave function b with values in $[-\infty,+\infty)$, which is analytic on the interior of its domain. In view of (2.10) -which holds for $\mu > 0$- and the usual extension of Γ the domain of b is $(-1,+\infty)$. We can see that $b(0) = 0$, $b'(0) = \sqrt{\pi}/2$, so that in view of (2.24) and (2.25), $\alpha(\lambda)$ is the supremum of those $\alpha \in (-1/2,+\infty) \cap (-1/2-\lambda,+\infty)$ such that

$$b(2(\alpha+\lambda)) + b(2\alpha) = 0. \tag{2.28}$$

Lemma 2.3 *i) The extended function b is one-to-one and $\mathcal{C}^\infty$ from $(-1,+\infty)$ to $(-\infty,+\infty)$. ii) For every $\lambda \in \mathbb{R}$ there exists a unique $\alpha(\lambda) \in (-1/2,+\infty) \cap (-1/2-\lambda,+\infty)$ such that (2.28) holds. Moreover α satisfies*

$$\alpha(-\lambda) = \alpha(\lambda) + \lambda\ . \tag{2.29}$$

iii) The function $\lambda \to \alpha(\lambda)$ is strictly decreasing from $\mathbb{R}$ onto $(-1/2,+\infty)$, continuously differentiable, with

$$\alpha'(\lambda) = -\frac{b'(2\alpha+2\lambda)}{b'(2\alpha)+b'(2\alpha+2\lambda)} \tag{2.30}$$

and α' is one-to-one from $\mathbb{R}$ to $(-1,0)$.

Proof:

i) Formulas (2.7)(2.8)(2.9) provide many properties of b as a log-Laplace transform of a positive random variable. In particular, b is strictly increasing, concave, $b(\mu) < 0$ if $\mu < 0$ and $b(\mu) > 0$ if $\mu > 0$, and $\lim_{\mu\to+\infty} b(\mu) = +\infty$.

ii) From i), the function $\alpha \mapsto b(2\alpha) + b(2\alpha+2\lambda)$ is one-to-one from $(-1/2,+\infty) \cap (-1/2-\lambda,\ +\infty)$ to $(-\infty,+\infty)$. (2.29) is an obvious consequence of the symmetry.

iii) Since $b' > 0$, (2.30) is a consequence of (2.28) and the implicit function Theorem. Again from (2.7)(2.8)(2.9), $b'' < 0$ and a short computation shows that $\alpha'' > 0$. In view of (2.28) we have $\lim_{\lambda\uparrow+\infty} \alpha(\lambda) = -1/2$ and $\lim_{\lambda\downarrow-\infty} \alpha(\lambda) = +\infty$. Moreover, in view of (2.8) the monotone convergence theorem yields $\lim_{\lambda\uparrow+\infty} b(\lambda)/\lambda = 0$ hence by concavity, $\lim_{\lambda\uparrow+\infty} b'(\lambda) = 0$ which allows to conclude $\alpha'(\mathbb{R}) = (-1, 0)$. $\square$

We remark that $\alpha(\lambda)$ is a pole of $\mathcal{L}^*$, hence

$$\mathcal{D}_\lambda = (\alpha(\lambda), +\infty)\,. \tag{2.31}$$

2.3 End of the proof of Theorem 1

To get an LDP for $\big(A^+(t)\big)$, we apply the Gärtner-Ellis theorem ([16] 2.3.6 p.44). The asymptotic behaviour of the Laplace transform of $A^+(t)$ is ruled by the following lemma.

Lemma 2.4 *For* $\lambda \in \mathbb{R}$,

$$\lim_{t\to\infty} e^{-t\alpha(\lambda)} E^* e^{-\lambda A^+(t)} = c(\lambda) := \frac{1}{4\sqrt{\pi}} \frac{\left[\dfrac{b(2\alpha)}{\alpha} + \dfrac{b(2\alpha+2\lambda)}{\alpha+\lambda}\right]^2}{b'(2\alpha) + b'(2\alpha+2\lambda)}\,. \tag{2.32}$$

From (2.32) we see that the limit

$$\Lambda(-\lambda) := \lim_{t\to\infty} \frac{1}{t} \log E^* e^{-\lambda A^+(t)} \tag{2.33}$$

exists for every $\lambda \in \mathbb{R}$ and satisfies

$$\Lambda(\lambda) = \alpha(-\lambda) = \lambda + \alpha(\lambda)\,. \tag{2.34}$$

In view of Lemma 2.3, Λ is finite and differentiable everywhere, so Theorem 2.3.6 of [16] states that the LDP holds with rate function

$$\mathbf{J}(x) = \sup_{\lambda\in\mathbb{R}} \{\lambda x - \Lambda(\lambda)\}. \tag{2.35}$$

Actually, again by Lemma 2.3, Λ is increasing from $-1/2$ to $+\infty$ with an asymptote $\lambda - 1/2$. Moreover,

$$\Lambda'(\lambda) = \frac{b'(2\alpha)}{b'(2\alpha) + b'(2\alpha+2\lambda)}. \tag{2.36}$$

Therefore

$$J(a) = a\lambda - (\alpha + \lambda)\,, \tag{2.37}$$

where λ is such that

$$a = \frac{b'(2\alpha)}{b'(2\alpha) + b'(2\alpha+2\lambda)}. \tag{2.38}$$

The dual J is a positive convex function, whose domain is $[0,1]$, vanishing only in $1/2$ and satisfying $J(0) = J(1) = 1/2$ with vertical tangents. These two values may be found otherwise considering the hitting time of 0, since

$$P^*(A_t^+ = t) = P^*(T_0 \geq t, X_0 > 0) \tag{2.39}$$

$$= \frac{1}{2} P(A_0 \leq e^{-t}) \text{ from (2.23)} \tag{2.40}$$

$$\sim \frac{1}{\pi} e^{-t/2} \text{ as } t \to \infty. \quad \square \tag{2.41}$$

<u>Proof of lemma 2.4</u>

In view of the above remarks on the symmetry between λ and $\alpha + \lambda$, it is enough to fix $\lambda < 0$, in which case $Ee^{-\lambda A^+(t)}$ is an increasing function of t.

From (2.24), (2.25) and (2.10) we have

$$\mathcal{L}^*(\lambda, \alpha) = H(\alpha) + \frac{K(\alpha)}{b(2\alpha) + b(2\alpha + 2\lambda)} \tag{2.42}$$

where H and K are holomorphic on $\Re\alpha > -1/2 - \lambda$. So, if $s > 0$, $\theta \in \mathbb{R}$,

$$\lim_{s \downarrow 0} \mathcal{L}^*(\lambda, \alpha(\lambda) + s + i\theta) - \frac{c(\lambda)}{s} = \phi(\theta) \tag{2.43}$$

uniformly in every finite interval $|\theta| \leq d$. We may apply the Tauberian theorem of Ikehara ([24], see [36] for a generalised formulation and [23] Remark p.397):

Theorem 2.5 (Ikehara) *If $M(t), t \in [0, \infty)$ is a non-negative non-decreasing function such that the integral*

$$F(\alpha) = \int_0^\infty e^{-\alpha t} M(t) dt \quad (\alpha = s + i\theta)$$

converges for $s > a > 0$, and if for some constant C and some function $g(\theta)$,

$$\lim_{s \downarrow a} F(\alpha) - \frac{C}{\alpha - a} = g(\theta)$$

uniformly in every finite interval $|\theta| \leq d$, then

$$\lim_{t \to \infty} M(t) e^{-at} = C \ . \quad \square$$

3 Empirical distribution and process level large deviations

If Q is a stationary distribution on $\Omega = \mathcal{C}([0,\infty), \mathbb{R})$ let us denote by $q(Q) \in M_1(\mathbb{R})$ its marginal. If

$$R_t := \frac{1}{t} \int_0^t \delta_{X_{s+\cdot}} ds,$$

then

$$q(R_t) = L_t := \frac{1}{t} \int_0^t \delta_{X_s} ds \quad \text{and} \quad q(R_t)(0, \infty) = \frac{A_t^+}{t},$$

so it is natural to consider the corresponding LDP for $(L_t, t > 0)$ and for $(R_t, t > 0)$.

When the underlying process is OU, we know from [11] exercise 5.4.40 p.230, that $(R_t, t > 0)$ satisfies a LDP for the τ topology, with the entropy $\mathbf{H}$ as rate function and that $(L_t, t > 0)$ satisfies a LDP with rate function defined for $\mu \in M_1(\mathbb{R})$ by

$$\left\{ \begin{array}{lcl} \mathbf{I}(\mu) & = & \frac{1}{2}\int (g')^2 d\mathcal{N} \;\text{ if } d\mu = g^2 d\mathcal{N} \\ & = & \infty \;\text{ otherwise.} \end{array} \right. \tag{3.1}$$

Let us precise the notion of entropy. Fix a probability measure R on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ which is translation invariant. We denote by $R_{0,\omega(0)}$ a regular conditional probability of R, given $\omega(0)$, and $R_{0,\omega}$ a regular conditional probability of R, given the whole past . The entropy $\mathbf{H}$ may be written as

$$\mathbf{H}(R) = E^R\Big[E^{R_{0,\omega}}\big[\log \frac{dR_{0,\omega(0)}}{dP_{0,\omega(0)}}|_{\mathcal{F}_1^0}\big]\Big]. \tag{3.2}$$

where $\mathcal{F}_1^0$ is the σ-field generated by the coordinates $\{\omega(s), s \in [0,1]\}$.

By the contraction principle we have

$$\mathbf{I}(\mu) = \inf\{\mathbf{H}(Q) \;\; ; \; q(Q) = \mu\}, \tag{3.3}$$

$$\mathbf{J}(a) = \inf\{\mathbf{H}(Q) \; ; \; q(Q)(0,\infty) = a\}. \tag{3.4}$$

and

$$\mathbf{J}(a) = \inf\{\mathbf{I}(\mu) \;\; ; \; \mu(0,\infty)) = a\} \tag{3.5}$$

By a direct computation, it can be seen that $\mathbf{I}$ is strictly convex. If we exhibit $\mu_a \in M_1(\mathbb{R})$ such that $\mathbf{I}(\mu_a) = \mathbf{J}(a)$, it will be the unique minimum of (3.5). The variational problem (3.5) is classical, but for the reader's convenience we give some details.

If we introduce the Lagrange multipliers $\lambda_\pm$, we need to study the variation of

$$\frac{1}{2}\int (g')^2 d\mathcal{N} + \lambda_+ \int_{\mathbb{R}^+} g^2 d\mathcal{N} + \lambda_- \int_{\mathbb{R}^-} g^2 d\mathcal{N}.$$

The infinitesimal generator of the OU process

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2} - \frac{x}{2}\frac{d}{dx},$$

and if $g \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\backslash\{0\})$,

$$\frac{1}{2}\int (g')^2 d\mathcal{N} = - < g, \mathcal{G}g >_{L^2(\mathcal{N})} . \tag{3.6}$$

We can then look for a critical $g \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\backslash\{0\})$, solving

$$\mathcal{G}g = hg \;\;\text{where}\;\; h = \lambda_\pm 1_{\mathbb{R}_\pm}, \tag{3.7}$$

and satisfying

$$\int_{\mathbb{R}^+} g^2 d\mathcal{N} = a \ , \ \int_{\mathbb{R}^-} g^2 d\mathcal{N} = 1 - a. \tag{3.8}$$

By convexity it will be such that

$$\mathbf{J}(a) = \frac{1}{2}\int (g')^2 d\mathcal{N} \ ; \ \int_{\mathbb{R}^+} g^2 d\mathcal{N} = a \ , \ \int_{\mathbb{R}^-} g^2 d\mathcal{N} = 1 - a. \tag{3.9}$$

Since we need $g \in L^2(\mathcal{N})$, the only possibility is

$$g(x) = M_+ H_{-2\lambda_+}(\frac{x}{\sqrt{2}}) \text{ for } x > 0 \tag{3.10}$$

$$g(x) = M_- H_{-2\lambda_-}(\frac{-x}{\sqrt{2}}) \text{ for } x < 0. \tag{3.11}$$

where H_ν is the Hermite function of index ν (see Erdelyi [17] and Lebedev [27] for the properties of these functions, and our appendix for a definition).

Since we are looking for g which is $\mathcal{C}^1$ in 0, we need the conditions:

$$M_+ H_{-2\lambda_+}(0) = M_- H_{-2\lambda_-}(0) \ \ , \ \ M_+ H'_{-2\lambda_+}(0) = -M_- H'_{-2\lambda_-}(0) \tag{3.12}$$

Actually,

$$\frac{H'_{-\theta}(0)}{H_{-\theta}(0)} = -2b(\theta) \tag{3.13}$$

and (3.12) entails in particular

$$b(2\lambda_+) + b(2\lambda_-) = 0. \tag{3.14}$$

Saturating the constraints (3.8) gives

$$M_+^2 \int_0^\infty \Big[H_{-2\lambda_+}(\frac{x}{\sqrt{2}})\Big]^2 d\mathcal{N}(x) = a \, , \ M_-^2 \int_{-\infty}^0 \Big[H_{-2\lambda_-}(\frac{-x}{\sqrt{2}})\Big]^2 d\mathcal{N}(x) = 1 - a \tag{3.15}$$

We know from Erdelyi ([17] p.122) that

$$\int_0^{+\infty} e^{-z^2/2} \left[H_\nu\left(\frac{z}{\sqrt{2}}\right)\right]^2 dz = 2^\nu \sqrt{\pi} 2^{-3/2} \frac{G(-\nu)}{\Gamma(-\nu)} \, .$$

where G is defined in Erdelyi [17] Chap I.7.:

$$G(\mu) := \Big(\log \Gamma\Big)'(\frac{\mu + 1}{2}) - \Big(\log \Gamma\Big)'(\frac{\mu}{2}). \tag{3.16}$$

Therefore we obtain

$$M_+^2 \frac{2^{-2\lambda_+ - 3/2}}{\sqrt{2}} \sqrt{\pi} \ \frac{G(2\lambda_+)}{\Gamma(2\lambda_+)} = a \, , \ M_-^2 \frac{2^{-2\lambda_- - 3/2}}{\sqrt{2}} \sqrt{\pi} \ \frac{G(2\lambda_-)}{\Gamma(2\lambda_-)} = 1 - a. \tag{3.17}$$

From formulas (3.12), (3.15) and the duplication formula for the Gamma function, we get

$$\frac{G(2\lambda_+)}{G(2\lambda_-)} = -\frac{a}{1-a}\,, \tag{3.18}$$

which fits with equation (2.38). Therefore, using (3.6) and (3.7), we obtain

$$\mathbf{I}(\mu_a) = a(\lambda_- - \lambda_+) - \lambda_-\,. \tag{3.19}$$

These two last formulas fit with (2.36) and (2.37) if we choose $\lambda_+ = \alpha$ and $\lambda_- = \alpha + \lambda$.

So far we have proved that $\mathbf{J}(a) = \mathbf{I}(\mu_a)$, where $d\mu_a(x) = g^2(x)d\mathcal{N}(x)$ and g is given by (3.10).

Moreover if $2\lambda_+$ and $2\lambda_-$ are so chosen, they are greater than -1, and g is strictly positive on $[0,\infty)$ (see for instance [17] II p.126 or [27] problem 1 p.297). On the contrary if $2\lambda_+$ for instance is chosen in $(-\infty,-1]$ then g has exactly one zero on $[0,\infty)$.

We can exhibit a stationary distribution Q such that $\mathbf{H}(Q) = \mathbf{I}(\mu_a) = \mathbf{J}(a)$. It is now clear, thanks to (3.12) that $\left(g(X_t)e^{-\int_0^t h(X_s)ds},\ \ t \geq 0\right)$ is a P_x non-negative martingale for every x. We can define as in [11] Lemma 5.3.5,

$$Q_x|_{\mathcal{F}_t} = \frac{g(X_t)}{g(x)}e^{-\int_0^t h(X_s)ds}\ .P_x|_{\mathcal{F}_t} \tag{3.20}$$

and under Q_x,

$$X_t = x + B_t - \frac{1}{2}\int_0^t X_s ds + \int_0^t \frac{g'}{g}(X_s)ds. \tag{3.21}$$

This diffusion is ergodic and its invariant measure is precisely μ_a. From (3.2) and (3.20) we get $\mathbf{H}(Q) = \mathbf{J}(a)$ (notice that $g^2 \log g \in L^1(\mathcal{N})$). We prove now that it is the unique probability with marginal μ_a satisfying this relation.

Assume that R is another probability. Following the notations of [15] and taking P for P^*, we have

$$H_P(R) = H_Q(R) + E^R\Big[E^{R_{0,\omega}}\big[\log\frac{dQ_{0,\omega(0)}}{dP_{0,\omega(0)}}|_{\mathcal{F}_1^0}\big]\Big] \tag{3.22}$$

From (3.20) again and the definition of h we have

$$\log\frac{dQ_{0,\omega(0)}}{dP_{0,\omega(0)}}|_{\mathcal{F}_1^0} = \log g(X_1) - \log g(X_0) - \lambda_+\int_0^1 1_{X_s>0}ds - \lambda_-\int_0^1 1_{X_s<0}ds \tag{3.23}$$

so that, by integration, using the stationarity of R

$$E^R\Big[E^{R_{0,\omega}}\big[\log\frac{dQ_{0,\omega(0)}}{dP_{0,\omega(0)}}|_{\mathcal{F}_1^0}\big]\Big] = -\lambda_+ a - \lambda_-(1-a) = J(a). \tag{3.24}$$

This entails that $H_Q(R) = 0$ i.e. $Q = R$.

We describe this diffusion, which satisfies the SDE (3.21), more completely. Its generator is

$$\tilde{\mathcal{G}} = \mathcal{G} + \frac{g'}{g}\frac{d}{dx}.$$

The process $M_t := e^{-(\lambda_+\tau_t^+ + \lambda_-\tau_t^-)}, t \geq 0$ is a $(\mathcal{F}_{\tau_t})$ martingale (with respect to P) and

$$Q|_{\mathcal{F}_{\tau_t}} = M_t.P|_{\mathcal{F}_{\tau_t}} \tag{3.25}$$

In terms of the excursion measures this gives

$$n_Q(d\epsilon) = n_P(d\epsilon)e^{-(\lambda_+ V_+(\epsilon)+\lambda_- V_-(\epsilon))}. \tag{3.26}$$

for, if F is any positive function,

$$\begin{aligned} E_Q\big[\sum_{s\leq t} F(X_{\tau_s^- + u},\ u \leq \tau_s - \tau_s^-)\big] &= E_P\Big[\exp - \sum\{...\}\ \exp\big(-(\lambda_+\tau_t^+ + \lambda_-\tau_t^-)\big)\Big] \\ &= \exp -t\int n_P(d\epsilon)\Big[1 - e^{-(F(\epsilon)+\lambda_+ V_+(\epsilon)+\lambda_- V_-(\epsilon))}\Big] \end{aligned}$$

and since M_t is a martingale, $EM_t = 1$ so $\int n_P(d\epsilon)\big(1 - e^{\left(\lambda_+ V_+(\epsilon)+\lambda_- V_-(\epsilon)\right)}\big) = 0$ which proves (3.26).

This yields in particular

$$\begin{cases} \tilde{E}\exp\big[-\mu\tau_t\big] = \exp -t\tilde{b}(\mu), & \tilde{b}(\mu) = b(2\lambda_+ + \mu) + b(2\lambda_- + \mu), \\ \tilde{E}e^{-\mu A^{\pm}_{\tau_t}} = \exp -t\tilde{b}_{\pm}(\mu), & \tilde{b}_{\pm}(\mu) = b(2\lambda_{\pm} + \mu) - b(2\lambda_{\pm}) \end{cases} \tag{3.27}$$

Let $n_V^{\pm}$ be the "distribution" under n of the length of $\pm$-excursions and put a tilde for the new process. The above result means that

$$\tilde{n}_V^{\pm}(ds) = \exp(-\lambda_{\pm}s)n_V^{\pm}(ds). \tag{3.28}$$

Let us now explain (3.18) with probabilistic arguments. We have Q a.s. as $t \to \infty$

$$\lim\ \frac{A^+_{\tau_t}}{t} = \tilde{b}'_+(0)\ ,\ \lim\ \frac{\tau_t}{t} = \tilde{b}'(0)$$

so,

$$\lim\ \frac{A^+_t}{t} = \lim\ \frac{A^+_{\tau_t}}{\tau_t} = \frac{\tilde{b}'_+(0)}{\tilde{b}'(0)} = \frac{b'(2\lambda_+)}{b'(2\lambda_+) + b'(2\lambda_-)}.$$

4 Extension to Walsh processes

Let $k \geq 2$, E_k, the union of k rays in $\mathbb{R}^2$ emerging from 0 with angles $\{\theta_1, \theta_2, .., \theta_k\}$ and $p = (p_i)_{1\leq i\leq k}$ a probability. We replace the linear BM by the Walsh process $(X_t = (R_t, \Theta_t))$ of index ν, with angular distribution p (see e.g. [2], [3] for details). We assume that $X_0 = 0$ and that (X_t) behaves as a Bessel process of dimension δ, $0 < \delta < 2$ on each ray while, upon arrival at 0, it chooses the $i-th$ ray with probability p_i ($\sum_i p_i = 1$); more precisely, (X_t) may be described by gathering its excursions away from 0 under the assumption that its Ito excursion measure is obtained from that of the BES^δ process weighting the i-th ray with the coefficient p_i.

Since the Bessel process has the Brownian scaling property, a strong arc-sine law similar to (1.1) can be proved:

$$S_T^{(i)} := \frac{1}{\log T} \int_1^T 1_{(\Theta_u = i)} \frac{du}{u} \to p_i \quad (T \to \infty) \quad \text{a.s. ,} \tag{4.1}$$

for $i = 1, .., k$.

It is known that the process $\bar{R}$ defined by

$$\bar{R}(t) = e^{-2ct} R^2(e^{2ct}/2c) \quad (-\infty < t < \infty)$$

is a two sided stationary process satisfying the SDE

$$d\bar{R}_t = (\delta - 2c\bar{R}_t)dt + 2\sqrt{\bar{R}_t}\, dW_t. \tag{4.2}$$

It is called a squared δ dimensional radial OU process with drift parameter c. It is ergodic and its stationary distribution is a gamma distribution. Let us fix $c = 1/2$ as in the preceding paragraphs and let us denote $\bar{\Theta}_t = \Theta_{e^t}$ and $\bar{X}_t = (\bar{R}_t, \bar{\Theta}_t)$.

We now extend the notations of Section 2.1.. For $A_t = \sum_{i=1}^k \lambda_i \int_0^t 1_{(\bar{\Theta}_s = i)} ds$ and

$$\mathcal{L}(\lambda_1, .., \lambda_k; \alpha) = E\Big[\int_0^\infty dt\ e^{-\alpha t - A_t}\Big],$$

we find $\mathcal{L} = \frac{N}{D}$ where

$$N = \int n_X(d\epsilon) \int_0^{V(\epsilon)} du\ e^{-\alpha u - A_u} = \sum_i p_i \int n_\nu(d\epsilon) \int_0^{V(\epsilon)} du\ e^{-(\alpha + \lambda_i)u} \tag{4.3}$$

$$D = \int n_X(d\epsilon) \left(1 - e^{-(\alpha V(\epsilon) + A_{V(\epsilon)})}\right) = \sum_i p_i \int n_\nu(d\epsilon) \Big(1 - e^{-(\alpha + \lambda_i)V(\epsilon)}\Big) \tag{4.4}$$

with n_ν the excursion measure of the squared OU$^\delta$ ([3] Lemme p.309), where $\nu = 1 - \frac{\delta}{2}$ $(0 < \nu < 1)$. This entails

$$N = \sum_i p_i \frac{b_\nu(\alpha + \lambda_i)}{\alpha + \lambda_i} \quad , \quad D = \sum_i p_i\ b_\nu(\alpha + \lambda_i) \tag{4.5}$$

where $b_\nu(a)$ is the Lévy exponent:

$$E\Big[\exp\Big(-a\tau_t\Big)\Big] = \exp\big(-t b_\nu(a)\big). \tag{4.6}$$

It remains to find an expression for b_ν. From [31], formula 16 or 59 (but changing a multiplicative constant owing to our convention (2.5) for the local time) we know that

$$n_\nu(V > s) = \frac{1}{\Gamma(1-\nu)} \frac{e^{-s\nu}}{(1 - e^{-s})^\nu} \tag{4.7}$$

so that

$$b_\nu(a) = \int_0^\infty a e^{-as} n_\nu(V > s) ds = \frac{\Gamma(a + \nu)}{\Gamma(a)}\ . \tag{4.8}$$

Finally, we have obtained

$$\mathcal{L} = \frac{\displaystyle\sum_i p_i \frac{\Gamma(\alpha+\lambda_i+\nu)}{(\alpha+\lambda_i)\Gamma(\alpha+\lambda_i)}}{\displaystyle\sum_i p_i \frac{\Gamma(\alpha+\lambda_i+\nu)}{\Gamma(\alpha+\lambda_i)}}\ , \tag{4.9}$$

(we find (2.13) If $\delta = 1, k = 2, p_1 = p_2 = 1/2, \lambda_1 = \lambda, \lambda_2 = 0$).

If P^* denotes the law of the squared OU^δ under its stationary distribution, one finds using Example 7 of [31] that $f^*(\alpha) := E^*(e^{-\alpha T_0}) = \frac{b_\nu(\alpha)}{\alpha\Gamma(\nu)}$ (this is a generalisation of Proposition 2.1). We may extend the method of section 2. Denoting

$$\Lambda(\lambda_1, ..., \lambda_k) = \lim_{t\to\infty} \frac{1}{t} \log E^* \exp\Big(\sum_{i=1}^k \lambda_i \int_0^t 1_{(\Theta_s = i)}\ ds\Big) \tag{4.10}$$

then $\Lambda(\lambda_1, ..., \lambda_k)$ is the unique solution α of

$$\sum_{i=1}^k p_i b_\nu(\alpha - \lambda_i) = 0. \tag{4.11}$$

such that $\alpha - \lambda_i > -1$ for every $i = 1, ..., k$.

For the same reasons as above, if we fix positive numbers a_i $i = 1, .., n$, such that $a_1 + .. + a_n = 1$, there exists a unique $(\lambda_1, ..., \lambda_n)$ such that

$$a_i = \frac{p_i b'_\nu(\alpha - \lambda_i)}{p_1 b'_\nu(\alpha-\lambda_1) + ... + p_n b'_\nu(\alpha - \lambda_n)}, \quad i = 1, .., n,$$

and the rate function is

$$J(a_1, ..., a_n) = (\lambda_1 a_1 + ... + \lambda_n a_n) - \alpha. \tag{4.12}$$

In the case $2 > \delta > 1$ we may use the OU^δ process (not squared) satisfying the SDE

$$dU_t = \Big(\frac{\delta - 1}{2U_t} - \frac{U_t}{2}\Big)dt + dW_t$$

and look at the higher levels of large deviations.

We may build a Walsh diffusion whose infinitesimal generator admits as restriction on the i-th ray

$$\mathcal{G}_i = \frac{1}{2}\frac{d}{dx^2} + \big(\frac{\delta-1}{2x} - \frac{x}{2} + \frac{g'_i}{g_i}\big)\frac{d}{dx} \tag{4.13}$$

where g_i is from $(0, \infty)$ to $(0, \infty)$ and satisfy $\mathcal{G} g_i = (\alpha - \lambda_i) g_i$, and the boundary condition is

$$p_1 g'_1(0) + ... + p_n g'_n(0) = 0.$$

Actually, the change of probability is given by a spider-martingale in the sense of [41].

Analogs of the last formulas of the above section hold also.

5 Partial-sum processes

Let $(X_k, k \geq 1)$ i.i.d. random variables and $S_k = X_1 + ... + X_k$. Erdös-Hunt [18] proved that if the distribution of X_1 is symmetric then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{(S_k>0)} = \frac{1}{2} \quad \text{a.s..} \tag{5.1}$$

It can be seen as a discrete version of the strong arc sine theorem.

Actually, the so-called Almost Everywhere Central Limit Theorem says that if $EX = 0$ and $EX_1^2 = 1$ then $L_n := \frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\delta_{S_k/\sqrt{k}}$ converge weakly to $\mathcal{N}$ ([7], [35] [26]).

Now, under a moment assumption, from Theorem 1,2,3,4 of [30], the sequence (L_n) satisfies a LDP at speed $(\log n)^{-1}$ and its rate function is the same as for the BM-OU process. It is proved by a strong approximation argument (Skorokhod representation theorem). Moreover, the result holds also at the process level. So we can assert :

Corollary 5.1 *If $E|X_1|^m < \infty$ for all $m > 0$, then $\left(\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k} 1_{(S_k>0)},\ n \geq 1\right)$ satisfies a LDP at speed $(\log n)^{-1}$ and with rate function $\mathbf{J}$ as defined in Theorem 1.1.*

6 Appendix

Let us give a complete expression of the Hermite functions:

$$\begin{aligned} H_{-\theta}(x) &= 2^{-\theta}\sqrt{\pi}\Big\{\frac{1}{\Gamma((\theta+1)/2)}\Big(1+\sum_{k=1}^{\infty}\frac{\theta(\theta+2)...(\theta+2k-2)}{3.5...(2k-1)}\frac{x^{2k}}{k!}\Big) \\ &- \frac{2x}{\Gamma(\theta/2)}\Big(1+\sum_{k=1}^{\infty}\frac{(\theta+1)(\theta+3)...(\theta+2k-1)}{3.5....(2k+1)}\frac{x^{2k}}{k!}\Big)\Big\} \end{aligned}$$

They are connected to the family of parabolic cylinder function ([17] Vol. II) by

$$D_\theta(z) = 2^{-\theta/2} e^{-z^2/4} H_\theta\left(\frac{z}{\sqrt{2}}\right). \tag{6.1}$$

For the OU process,

$$f(x,\alpha) = \frac{e^{\frac{x^2}{4}} D_{-2\alpha}(|x|)}{D_{-2\alpha}(0)} = \frac{H_{-2\alpha}\left(\frac{|x|}{\sqrt{2}}\right)}{H_{-2\alpha}(0)} \tag{6.2}$$

(recall that $f(x,\alpha) = E_x e^{-\alpha T_0}$).

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