ON TWO MEASURES DEFINED ON THE BOUNDARY OF A BRANCHING TREE

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Abstract: Replying to an question of A. Joffe, we show that two random measures defined on the boundary of a Galton-Watson tree are mutually singular. We compare them in a precise way, and we extend this result to marked trees in the framework of random fractals.

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1. Introduction and Notations

In order to define the two mesures on the boundary of a Galton-Watson tree, let us recall some definitions, using for trees the notations of Neveu (1986). Let $\mathbf{N}^\star$ be the set of positive integers, $(\mathbf{N}^\star)^k$ the set of all k term sequences, $\mathbf{U} = \bigcup_{k=0}^{\infty} (\mathbf{N}^\star)^k$ the set of all finite sequences and $\mathbf{I} = \mathbf{N}^{\star\mathbf{N}^\star}$ the set of infinite sequences $\mathbf{i} = (i_1, i_2, ..)$. We make the convention that $\mathbf{N}^{\star 0}$ contains the null sequence $\emptyset$. If $\mathbf{i} = (i_1, i_2, ..i_n)$ $(n \le \infty)$ is a sequence, we write $|\,\mathbf{i}\,| = n$ for its length, and $\mathbf{i}\,|\,k = (i_1, i_2, ..i_k)$ $(k \le n; \mathbf{i}\,|\,0 = \emptyset)$ for the curtailment of $\mathbf{i}$ after k terms. If $\mathbf{i} \in \mathbf{U}$ and $\mathbf{j} \in \mathbf{U}$ or $\mathbf{I}$ we write $\mathbf{ij}$ for the sequence obtained by juxtaposition. We partially order $\mathbf{U}$ by writing $\sigma < \tau$ to mean that the sequence τ is an extension of the sequence σ. We use a similar notation if $\sigma \in \mathbf{U}$ and $\tau \in \mathbf{I}$. If $\mathbf{i}$ and $\mathbf{j}$ are two sequences, we write $\mathbf{i} \wedge \mathbf{j}$ for the common maximal sequence of $\mathbf{i}$ and $\mathbf{j}$, that is, the maximal sequence $\mathbf{q}$ such that $\mathbf{q} < \mathbf{i}$ and $\mathbf{q} < \mathbf{j}$.

A tree ω is a subset of $\mathbf{U}$ satisfying three conditions:

i) $\emptyset \in \omega$

ii) $\mathbf{i} \in \omega$ implies $\mathbf{i}' \in \omega$ for all $\mathbf{i}' < \mathbf{i}$

iii) If $\mathbf{i} \in \omega$ and $j \in \mathbf{N}^\star$, then $\mathbf{i}j \in \omega$ if and only if $1 \le j \le N_{\mathbf{i}}$ for a positive integer $N_{\mathbf{i}}$.

An element of ω is called a node. We write $z_k(\omega)$ for the set of sequences in ω of length k and $Z_k(\omega)$ for the cardinal of $z_k(\omega)$. If ω is infinite, we define $\partial\omega$ as the boundary of ω. It is the set of infinite sequences $\mathbf{j}$ such that $\mathbf{i} \in \omega$ for every finite curtailment $\mathbf{i} < \mathbf{j}$. It is also called the branching set associated with ω (Hawkes 1981). We may regard $\partial\omega$ as a topological space in a natural way by taking as a basis $\{B(\mathbf{i})\}_{\mathbf{i}\in\omega}$ where

$$B(\mathbf{i}) = \{\mathbf{j} \in \partial\omega : \mathbf{i} < \mathbf{j}\}.$$

With this topology, $\partial\omega$ is metrizable and compact; a possible choice of the metric is $d_c(\mathbf{i}, \mathbf{j}) = c^{-|\mathbf{i}\wedge\mathbf{j}|}$ where $c > 1$ is given (Liu 1992). Actually we will use $c = e$ to get Neperian log in the Hausdorff dimension results. Then, $B(\mathbf{i})$ is a ball of radius $e^{-|\mathbf{i}|}$. Let Ω be the set of all trees ω and for every $n \in \mathbf{N}$, let $\mathcal{F}_n = \sigma(N_{\mathbf{i}} : |\,\mathbf{i}\,| < n)$ and finally let $\mathcal{F} = \sigma(\mathcal{F}_n, n \in \mathbf{N})$.

Giving a distribution p on $\mathbf{N}$, there is a unique probability P on $(\Omega, \mathcal{F})$ satisfying the branching property and such that the offspring distribution of a node is p. We assume that p satisfies $p_0 = 0$, $p_k < 1$ for every $k \ge 1$ and that $m := \sum_{n=1}^{\infty} kp_k < \infty$. The branching process is supercritical and a tree ω is P-a.s. infinite.

It is known (Athreya-Ney Th.3 p.30) that there exists a deterministic sequence C_n and a random variable W a.s. strictly positive such that as $n \to \infty$,

$$C_{n+1}/C_n \to m \quad \text{and} \quad Z_n/C_n \to W \quad \text{a.s..}$$

Moreover, $C_n = m^n$ if and only if $E(N \log N) < \infty$. For every $\mathbf{i} \in \omega$, let $\Theta_{\mathbf{i}}$ the shift on the tree at the node $\mathbf{i}$. The sequence $Z_n \circ \Theta_{\mathbf{i}}/C_n$ converges a.s. to a random variable $W_{\mathbf{i}}$. From

$$Z_n \circ \Theta_{\mathbf{i}} = \sum_{\mathbf{j}\in z_k\circ\Theta_{\mathbf{i}}} Z_{n-k} \circ \Theta_{\mathbf{j}} \text{ for every } \mathbf{i} \in \omega, k < n \in \mathbf{N}^\star$$

we deduce easily:

$$W_{\mathbf{i}} = \sum_{\mathbf{j}\in z_k\circ\Theta_{\mathbf{i}}} m^{-k} W_{\mathbf{j}} \text{ for every } \mathbf{i} \in \omega, k \in \mathbf{N}^\star$$

For every ω, let μ^0_ω be the unique Borel (random) measure on $\partial\omega$ such that:

$$\mu^0_\omega(B(\mathbf{i}\,|\,n)) = m^{-n} W_{\mathbf{i}|n} \quad \text{for every } \mathbf{i} \in \partial\omega, n \in \mathbf{N} \quad (\text{with } \mu^0_\omega(B(\mathbf{i}\,|\,0)) = \mu^0_\omega(\partial\omega) = W).$$

For every ω, $\mu_\omega = \mu^0_\omega / W$ is then a (random) probability measure on $\partial\omega$. This measure does not depend upon the choice of the Seneta-Heyde norming. Let now for every ω, the Borel random measure ν_ω on $\partial\omega$ be such that:

$$\nu_\omega(B(\mathbf{i}|n)) = \prod_{k=0}^{n-1} \frac{1}{N_{\mathbf{i}|k}}; \quad \nu_\omega(B(\mathbf{i}|0)) = 1.$$

It is known that for P-almost every tree ω, μ_ω is non-atomic. This result is due to O'Brien (1980) (see also Joffe 1978). Joffe (1978) asked under which conditions μ could be absolutely continuous with respect to ν. We prove in Section 2 that they are a.s. mutually singular and we compare them in a precise way. Let us now give the meaning of these measures.

Let the tree ω be given. The measure ν_ω is easily understood as a downstream distribution of mass. We can imagine the ancestor owning an initial amount of money and dividing it in equal parts for each of his children, and so on.... The measure μ_ω is the weak limit of $\mu^n_\omega, n \leq 1$ which corresponds to an upstream distribution of mass starting from the uniform mass in the $n^{th}-$generation. With our notations, this yields, for $\mathbf{i} \in \partial\omega$:

$$\mu^n_\omega(B(\mathbf{i}|n)) = \frac{1}{Z_n}$$

$$\mu^n_\omega(B(\mathbf{i}|k)) = \frac{Z_{n-k} \circ \Theta_{\mathbf{i}|k}}{Z_n} \qquad k < n,$$

and for $k > n$, $\mu^n_\omega(B(\mathbf{i}|k))$ defined in any consistent way.

Just before presenting our results to the workshop, we read the preprint of Lyons, Pemantle and Peres (1994). There is some overlap (which will be precised in Section 2) between parts of our theorem 3 and their paper. They call ν the "visibility measure" denoted by VIS, and μ the "uniform limiting measure" denoted by UNIF. In Section 3 and 4, we extend our results of Section 2 to the case of marked trees. Each node is provided with a random positive mark and marks are multiplied along a branch. Passing to log, this is similar to the branching random walk model. We define $\overline{\mu}$ and $\overline{\nu}$, prove that they are mutually singular and give their Hausdorff dimensions. We may recover the Galton-Watson case, giving to all marks the (deterministic) value $(EN)^{-1}$.

2 - Galton-Watson trees

Theorem 1 - *For $P-a.e.tree\,\omega$, ν_ω is non-atomic and the measures μ_ω and ν_ω are mutually singular if and only if N is not a.s. constant. Otherwise they are identical.*

Proof of Theorem 1 - In the proof, we will use the following easy result (whose proof is omitted) twice.

Lemma 2: *For every $\alpha \geq 0$, the sequence $M^\alpha_n := \sum_{\mathbf{i}\in z_n}[\nu_\bullet(B(\mathbf{i}))]^\alpha / (\sum_k k^{1-\alpha} p_k)^n$ is a (positive) $\mathcal{F}_n$-martingale.*

a) Since M^2_n has an a.s. limit and $\sum_k k^{-1} p_k < 1$, the sequence $\sum_{\mathbf{i}\in z_n}[\nu_\bullet(B(\mathbf{i})]^2$ converges a.s. to 0, hence $\nu_\bullet$ is a.s. non-atomic.

b) To prove the second part, we use the Hellinger distance between μ_ω and ν_ω. For every n, the balls $B(\mathbf{i}), \mathbf{i} \in z_n$ form a partition of $\partial\omega$. The sequence $\rho_n(\omega),\ n \geq 1$ defined by

$$\rho_n(\omega) := \sum_{\mathbf{i}\in z_n} \sqrt{\mu_\omega(B(\mathbf{i}))\nu_\omega(B(\mathbf{i}))}$$

is non increasing. The two measures μ_ω and ν_ω are mutually singular if and only if the limit of $\rho_n(\omega)$ is 0 (see Dacunha-Castelle and Duflo Th. 2.5.21), or -and this is equivalent- if the limit of $\rho_n(\omega)\sqrt{W(\omega)}$ is 0. To prove this a.s. convergence, it is sufficient, again since $\rho_n(\omega)$ is non increasing, to prove a L^1-convergence. Now

$$E\big(\rho_n\sqrt{W}\big) = E\big[E^{\mathcal{F}_n}\sum_{\mathbf{i}\in z_n}\sqrt{W\mu_\bullet(B(\mathbf{i}))\nu_\bullet(B(\mathbf{i}))}\big] = E\big[\sum_{\mathbf{i}\in z_n}\sqrt{\nu_\bullet(B(\mathbf{i}))}\; m^{-n/2}\;\big(E^{\mathcal{F}_n}\sqrt{W^{\mathbf{i}}}\big)\big]$$

and from the branching property, we get, a.s. for every $\mathbf{i}\in z_n$,

$$E^{\mathcal{F}_n}\sqrt{W^{\mathbf{i}}} = E\sqrt{W},$$

(which is finite, see Athreya-Ney 1972, p.63 ex.9). Hence it is sufficient to prove that $m^{-n/2}E\sum_{\mathbf{i}\in z_n}\sqrt{\nu_\bullet(B(\mathbf{i}))}\to 0$ as $n\to\infty$. In view of lemma 2 it is then sufficient to prove that

$$m^{-1/2}\{\sum_k k^{1/2}p_k\} < 1,$$

If N is not a.s. constant, this is a consequence of the strict concavity of the fonction $x\mapsto\sqrt{x}$. If $N = m$ a.s. we get $W^{\mathbf{i}} = 1$ for every $\mathbf{i}$ and $\mu_\omega(B(\mathbf{i}|n)) = \nu_\omega(B(\mathbf{i}|n)) = m^{-n}$.

Under some simple and classical conditions on the moments, the above study is sharpened in the following theorem.

Theorem 3 -
a) For P-a.e.tree ω, for μ_ω- a.e. $\mathbf{i}\in\partial\omega$,

(1) $$\frac{\log\mu_\omega(B(\mathbf{i}|n))}{n}\to -\log(EN), \qquad \textit{if}\quad E(N\log N)<\infty.$$

(2) $$\frac{\log\nu_\omega(B(\mathbf{i}|n))}{n}\to -\frac{E(N\log N)}{EN}, \qquad \textit{if}\quad E(N\log N)<\infty,$$

b) For P-a.e.tree ω, for ν_ω- a.e. $\mathbf{i}\in\partial\omega$,

(3) $$\frac{\log\mu_\omega(B(\mathbf{i}|n))}{n}\to -\log(EN) \qquad \textit{if}\quad E(N\log N)<\infty,$$

(4) $$\frac{\log\nu_\omega(B(\mathbf{i}|n))}{n}\to -E(\log N).$$

Corollary 4 -
1) *If $E(N\log N)<\infty$, then for a.e. tree ω, μ_ω is carried by a Borel set of Hausdorff dimension $D_\mu := \log(EN)$, whereas every Borel set of dimension strictly lower than $D_\mu$has μ_ω- measure 0.*

2) *For a.e. tree* ω, ν_ω *is carried by a Borel set of Hausdorff dimension* $D_\nu := E(\log N)$, *whereas every Borel set of dimension strictly lower than* D_ν *has* ν_ω*- measure* 0. *(Notice that* $D_\nu < D_\mu$*).*

Remark 5 -

a) (1) was proved by Hawkes (1981) under the condition $E(N \log^2 N) < \infty$. Applying (5) of theorem 7 below, we could get the same result also under the same condition. Actually Lyons, Pemantle and Peres (1994) relaxed to $E(N \log N) < \infty$. Notice that (1) implies in particular that for a.e.ω and ν_ω- a.e.$\mathbf{i} \in \partial\omega$, $\mu_\omega(B(\mathbf{i} \mid n)) \to 0$, which is equivalent to non-atomicity of μ_ω.

b) In the same way, (4) implies that for a.e.ω, ν_ω is non-atomic.

c) Let

$$A_\omega = \{\mathbf{i} \in \partial\omega : \frac{\log \mu_\omega(B(\mathbf{i}|n)}{n} \to -\log EN)\},$$

and

$$B_\omega = \{\mathbf{i} \in \partial\omega : \frac{\log \nu_\omega(B(\mathbf{i}|n)}{n} \to -E(\log N)\}.$$

From (1), if $E(N \log N) < \infty$ we have a.s. $\mu_\omega(A_\omega) = 1$, and from (4), a.s. $\nu_\omega(B_\omega) = 1$. These two results are in the following corollary. On the other hand, if $E(N \log N) < \infty$, from (2) we have a.s. $\mu_\omega(B_\omega) = 0$, which implies that a.s. μ_ω and ν_ω are mutually singular.

We will prove (2) (3) and (4) of theorem 3 and its corollary in section 4, as consequences of the more general case of marked trees.

3 - Marked trees.

Hereafter we consider a space (still denoted by Ω) of trees marked in the following way. To every node $\mathbf{i}$, we assign $N_\mathbf{i}$ positive numbers: $T_{\mathbf{i}1}, \ldots, T_{\mathbf{i}N_\mathbf{i}}$. We consider the associated filtration defined by:

$$\mathcal{F}_1 = \sigma(N; T_1, .., T_N) \text{ and for every } \quad k \geq 0, \mathcal{F}_{k+1} = \sigma(\mathcal{F}_k; N_\mathbf{i}, T_{\mathbf{i}q}, 1 \leq q \leq N_\mathbf{i}, \mathbf{i} \in z_k)$$

and let $\mathcal{F} := \sigma(\mathcal{F}_k, k \geq 1)$. We provide $(\Omega, \mathcal{F})$ with a probability P satisfying the branching property (see Neveu 1986 for marked trees). In particular, for every k, conditionally on $\mathcal{F}_k$, the random vectors $\tau_\mathbf{i} := (N_\mathbf{i}; T_{\mathbf{i}1}, \ldots, T_{\mathbf{i}N_\mathbf{i}}), \mathbf{i} \in z_k$ are independent and distributed as $\tau := \tau_\emptyset = (N; T_1, \ldots, T_N)$. Let $S_\mathbf{i} := \sum_{k=1}^{N_\mathbf{i}} T_{\mathbf{i}k}$, and $S := S_\emptyset$. We denote again $\partial\omega$ as the boundary of the underlying tree (without marks). Let first $\overline{\nu}_\omega$ be the (unique) random measure on $\partial\omega$ such that $\overline{\nu}_\omega(B(\mathbf{i})) = \prod_{\mathbf{j}<\mathbf{i}} T'_\mathbf{j} := Y'_\mathbf{i}$, where $T'_{\mathbf{i}k} = \dfrac{T_{\mathbf{i}k}}{S_\mathbf{i}}$. (We can now imagine a sharing of money at each node $\mathbf{i}$ in (random) unequal parts among the $N_\mathbf{i}$ vertices.)

In order to define $\overline{\mu}$, notice that if $ES = 1$, the following limits

$$\overline{W}_\mathbf{i} := \lim_{n\to\infty} \sum_{|\mathbf{j}|=n} \prod_{k=1}^{|\mathbf{j}|} T_{\mathbf{i}(\mathbf{j}|k)}, \qquad \overline{W} := \overline{W}_\emptyset$$

exist by the branching property and the (positive) martingale convergence theorem. Actually, under the following set of assumptions:

(H) : $N > 0$ a.s., $ES = 1, \exists\delta : E(S\log_+^{1+\delta} S)) < \infty$ *and* $-\infty < E(\sum_{k=1}^N T_k \log T_k) < 0$.

Biggins (1977) proved that

$$(*) \qquad E\overline{W} = 1 \text{ and } \overline{W} > 0 \text{ a.s.}$$

under the additional assumption that $E\big(\sum_{k=1}^N T_k(\log^+ T_k)^2\big) < \infty$. Liu (1994) proved (*) under *(H)* and the additional condition $EN < \infty$. Lyons (1994) proved (*) under *(H)* without any additional condition.

Let then $\overline{\mu}_\omega^0$ be the (unique) random measure on $\partial\omega$ such that for $\mathbf{i} \in \omega, \overline{\mu}_\omega^0(B(\mathbf{i})) = X_\mathbf{i}\overline{W}_\mathbf{i}$, where $Y_\mathbf{i} := \prod_{\mathbf{j}<\mathbf{i}} T_\mathbf{j}$. The measure $\overline{\mu}_\omega := \frac{1}{\overline{W}}\overline{\mu}_\omega^0$ is then a probability. When $T_k = \frac{1}{m}$, we find again $\overline{\mu}_\omega = \mu_\omega$ and $\overline{\nu}_\omega = \nu_\omega$. The following theorems extend theorems 1 and 3.

Theorem 6 - *Assume (H) satisfied. For P-a.e.tree ω, the measures $\overline{\mu}_\omega$ and $\overline{\nu}_\omega$ are non-atomic. They are mutually singular if and only if* S *is not a.s.* 1, *and otherwise they are identical.*

Let

(H_1) : $E(\overline{W}\log^+\overline{W}) < \infty$,

(H_2) : $E(|\log\overline{W}|) < \infty$.

Theorem 7 - *For P-a.e.tree ω and $\overline{\mu}_\omega$- a.e. $\mathbf{i} \in \partial\omega$,*

$$(5) \qquad \frac{\log\overline{\mu}_\omega(B(\mathbf{i}|n)}{n} \to E(\sum_{k=1}^N T_k \log T_k), \qquad \textit{if} \quad (H) \textit{and} (H_1),$$

$$(6) \qquad \frac{\log\overline{\nu}_\omega(B(\mathbf{i}|n)}{n} \to E(\sum_{k=1}^N T_k \log T_k) - E(S\log S) \qquad \textit{if} (H)$$

For P-a.e.tree ω, for $\overline{\nu}_\omega$ - almost every $\mathbf{i} \in \partial\omega$,

$$(7) \qquad \frac{\log\overline{\mu}_\omega(B(\mathbf{i}|n)}{n} \to E(\frac{1}{S}\sum_{k=1}^N T_k \log T_k), \qquad \textit{if} \quad (H) \textit{and} (H_2),$$

$$(8) \qquad \frac{\log\overline{\nu}_\omega(B(\mathbf{i}|n)}{n} \to E(\frac{1}{S}\sum_{k=1}^N T_k \log T_k) - E(\log S).$$

Corollary 8 -

1) *If* (H) and (H_1), *then for P-a.e. tree ω, $\overline{\mu}_\omega$ is carried by a Borel set of Hausdorff dimension* $D_{\overline{\mu}} := -E\sum_{k=1}^N T_k \log T_k$, *and every Borel set of dimension strictly lower than* $D_{\overline{\mu}}$ *has $\overline{\mu}$ -measure 0.*

2) *For P-a.e. tree* ω, $\overline{\nu}_\omega$ *is carried by a Borel set of dimension:*

$$D_{\overline{\nu}} := -E(\sum_{k=1}^{N} \frac{T_k}{S} \log \frac{T_k}{S}),$$

and every Borel set of dimension strictly lower than $D_{\overline{\nu}}$ *has* $\overline{\nu}$ *-measure 0.*

The proof of corollary 8 is a modification of Billingsley's argument (1965, pp. 136−145) or of Hawkes (1981).

Remark 9 -
a) $E(S\log S) \geq ES\log ES = 0$ and $E(\log S) \leq \log ES = 0$. (With equalities if and only if $S = 1$ a.s..)
b) In the Galton-Watson case $D_\nu < D_\mu$ (cf. Corollary 4). Lyons, Pemantle and Peres (1994) conjecture that to any flow rule other than UNIF there corresponds a measure of dimension less than D_μ, and prove that this is true for a class of flow rules. Here is an example of a marked tree satisfying $D_{\overline{\mu}} < D_{\overline{\nu}}$. Let U a positive random variable satisfying $P(U = 0 \text{ or } 1 = 0)$ and $EU = 1$. Let N be integer valued and satisfying $\log N > \dfrac{1+U}{1-U}\log\dfrac{1}{U}$ on $0 < U < 1$ and $\log N < \dfrac{U+1}{U-1}\log U$ on $U > 1$. Then if $T_k = \frac{U}{N}$ for every $1 \leq k \leq N$ we get $D_{\overline{\mu}} = -EU\log\dfrac{U}{N}$ and $D_{\overline{\nu}} = E\log N$. It then easy to check that $D_{\overline{\mu}} < D_{\overline{\nu}}$.
c) (5) extends a result of Kahane and Peyrière (1976) for a turbulence model. The measures $\overline{\mu}$ and $\overline{\nu}$ were also used, directly or implicitly by Graf, Mauldin and Williams (1988), Mauldin and Williams (1986), Falconer (1986) and Liu (1992 et 1993) in the study of random fractals or flows in networks.
d) Assume $T_k = \frac{1}{m}$, hence $\overline{\mu}_\omega = \mu_\omega$ and $\nu_\omega = \nu_\omega$. Assumption (H) is then satisfied if and only if $E(N\log N) < \infty$. Theorem 6 gives again theorem 1, but it can be noticed that it holds without (H), using the Seneta-Heyde norming. This improvement seems difficult to extend in the general case. (6) gives (2) and (8) gives (4) On the other hand, since for a Galton-Watson process, assumption $E(\overline{W}\log^+\overline{W}) < \infty$ is equivalent to $E(N\log^2 N) < \infty$, (5) gives (1) but under a stronger assumption. At the end of this paper, we prove that Galton-Watson satisfies (H_2), hence (7) gives (3).
e) For a better understanding of corollary 8, we give now an example related to number theory. Let $(N; T_1, ..., T_N) = (r; p_1, ..., p_r)$ a constant vector with $r \geq 2$ and $\sum_{k=1}^r p_k = 1$. The corresponding branching set $\tilde{\omega}$ is then $\{1, ..., r\}^{\mathbf{N}^\star}$. It is easy to check that $\overline{\mu}$ and $\overline{\nu}$ are then identical and coincide with $p^{\mathbf{N}^\star}$. For every $l \in \{1, .., r\}$ fixed, the events $\{\mathbf{i}_k = l\} := \{\mathbf{i} = \mathbf{i}_1\mathbf{i}_2..\mathbf{i}_k.. \in \tilde{\omega} : \mathbf{i}_k = l\}$ for $k \in \mathbf{N}^\star$ are $\overline{\mu}$- independent and have same probability. The law of large numbers yields:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n 1_{\mathbf{i}_k = l} = p_l \quad \overline{\mu}-a.s.,\ l = 1, .., r.$$

Hence the measure $\overline{\mu}$ (or $\overline{\nu}$) is carried by the set

$$A = \{\mathbf{i} = \mathbf{i}_1\mathbf{i}_2..\mathbf{i}_k.. \in \tilde{\omega} : \forall l = 1, .., r,\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n 1_{\mathbf{i}_k = l} = p_l\}$$

of all infinite sequences $\mathbf{i}$ such that for every l, the proportion of digit l is asymptotically equal to p_l. It is easy to check that on A,

$$\lim_{n\to\infty}\frac{\log\overline{\mu}(B(\mathbf{i}|n))}{n} = \sum_{k=1}^r p_k\log p_k.$$

With the distance $d_r(\mathbf{i}, \mathbf{j}) = r^{-|\mathbf{i}\wedge\mathbf{j}|}$, A has dimension $D_{\overline{\mu}} = -\dfrac{\sum_{k=1}^r p_k\log p_k}{\log r}$. This is the classical result of Eggleston: the subset

$$A^0 := \{x = \sum_{k=1}^\infty x_k r^{-k} : \lim_{n\to\infty}\frac{\sum_{k=1}^n 1_{\mathbf{i}_k = l}}{n} = p_l, l = 0, .., r-1\}$$

of $[0,1]$ has dimension $D_{\overline{\mu}}$, see for instance Billingsley (1965, p.139). Notice that A^0 is image of A by the map:

$$\mathbf{i} = \mathbf{i}_1\mathbf{i}_2..\mathbf{i}_k.. \mapsto \sum_{k=1}^{\infty}(\mathbf{i}_k - 1)r^{-k}.$$

4. Proof of Theorems 6 and 7 and consequences

To prove these theorems we will define a new measure on a product space. This is a classical technique in random measures, similar to the construction of the Campbell measure for point processes. First we extend $\overline{\mu}_\omega$ on $\mathbf{N}^{\star\mathbf{N}^\star}$: for $A \subset \mathbf{N}^{\star\mathbf{N}^\star}$, we define $\overline{\mu}_\omega(A) = \overline{\mu}_\omega(A \cap \tilde{\omega})$. In the same way we can extend $\overline{\nu}_\omega$. We define the probability measures $Q_{\overline{\mu}}$ and $Q_{\overline{\nu}}$ on $\Omega \times \mathbf{N}^{\star\mathbf{N}^\star}$ by:

$$Q_{\overline{\mu}}(A) = E\int 1_A(\omega,\mathbf{i})\ d\overline{\mu}^0_\omega(\mathbf{i}) \ \text{ and }\ Q_{\overline{\nu}}(A) = E\int 1_A(\omega,\mathbf{i})\ d\overline{\nu}_\omega(\mathbf{i}).$$

Next we define $\widehat{T}_n(\omega,\mathbf{i}) = T_{\mathbf{i}|n}(\omega)$, $\widehat{T}'_n(\omega,\mathbf{i}) = T'_{\mathbf{i}|n}(\omega)$, $\widehat{W}_n(\omega,\mathbf{i}) = W_{\mathbf{i}|n}(\omega), n = 1,2,..$ and we write $\widehat{T} = \widehat{T}_1$, $\widehat{T}' = \widehat{T}'_1$, $\widehat{W} = \widehat{W}_1$. The following lemma gives the precise structure of these variables.

Lemma 10 - *The two sequences $(\widehat{T}_n; n \geq 1)$ and $(\widehat{T}'_n; n \geq 1)$ are composed of random variables which are i.i.d under $Q_{\overline{\mu}}$ and $Q_{\overline{\nu}}$. Moreover if f is a Borel positive fonction, then*

$$E_{Q_{\overline{\mu}}} f(\widehat{T}_n) = E(\sum_{k=1}^{N} T_k f(T_k)) \quad \textit{and} \quad E_{Q_{\overline{\nu}}} f(\widehat{T}_n) = E(\frac{1}{S}\sum_{k=1}^{N} T_k f(T_k)), \tag{9}$$

$$E_{Q_{\overline{\mu}}} f(\widehat{T}'_n) = E(\sum_{k=1}^{N} T_k f(\frac{T_k}{S})) \quad \textit{and} \quad E_{Q_{\overline{\nu}}} f(\widehat{T}'_n) = E(\sum_{k=1}^{N} \frac{T_k}{S} f(\frac{T_k}{S})), \tag{10}$$

$$E_{Q_{\overline{\mu}}} f(\widehat{W}_n) = E(\overline{W} f(\overline{W})) \quad \textit{and} \quad E_{Q_{\overline{\nu}}} f(\widehat{W}_n) = Ef(\overline{W}). \tag{11}$$

Proof of lemma 10 :

i) We will show that the random variables $\widehat{T}_n, (n \geq 1)$ are $Q_{\overline{\mu}}$ -i.i.d. and that for every Borel nonnegative function f : $E_{Q_{\overline{\mu}}} f(\widehat{T}_n) = E\sum_{k=1}^{N} T_k f(T_k)$.
From the definition of $Q_{\overline{\mu}}$ and the branching property, we have:

$$E_{Q_{\overline{\mu}}} f(\widehat{T}_n) = E\sum_{\mathbf{i}\in z_n} f(T_\mathbf{i}) X_\mathbf{i} \overline{W}_\mathbf{i} = E(\sum_{\mathbf{i}\in z_n} f(T_\mathbf{i}) X_\mathbf{i})\ E\overline{W}$$

$$= E\Big(\sum_{\mathbf{i}\in z_{n-1}} X_\mathbf{i}\ [\sum_{k=1}^{N_\mathbf{i}} f(T_{\mathbf{ik}}) T_{\mathbf{ik}}]\Big) = E[\sum_{\mathbf{i}\in z_{n-1}} X_\mathbf{i}]\ E\sum_{k=1}^{N} T_k f(T_k) = E\sum_{k=1}^{N} T_k f(T_k).$$

So $\widehat{T}_n, (n \geq 1)$ are identically distributed. For the sake of simplicity, we will show only the two-by-two independence . For f and g Borel positive functions and $n < l$, we have:

$$E_{Q_{\overline{\mu}}}[f(\widehat{T}_n)g(\widehat{T}_l)] = E\sum_{\mathbf{i}\in z_l} f(T_{\mathbf{i}|\mathbf{n}})g(T_{\mathbf{i}})X_{\mathbf{i}}\overline{W}_{\mathbf{i}}$$

$$= E[\sum_{\mathbf{i}\in z_l} f(T_{\mathbf{i}|\mathbf{n}})g(T_{\mathbf{i}})X_{\mathbf{i}}]\ E\overline{W} \qquad (E\overline{W}=1)$$

$$= E\big[\sum_{\mathbf{i}\in z_{l-1}} f(T_{\mathbf{i}|n})X_{\mathbf{i}}\big(\sum_{k=1}^{N_{\mathbf{i}}} g(T_{\mathbf{i}k})T_{\mathbf{i}k}\big)\big] = E[\sum_{\mathbf{i}\in z_{l-1}} f(T_{\mathbf{i}|\mathbf{n}})X_{\mathbf{i}}]\ E\sum_{k=1}^{N} T_k g(T_k)$$

$$= E[\sum_{\mathbf{i}\in z_n} f(T_{\mathbf{i}})X_{\mathbf{i}}]\ E\sum_{k=1}^{N} T_k g(T_k) \qquad (E\sum_{k=1}^{N} T_k = 1)$$

$$= E_{Q_{\overline{\mu}}}(f(\widehat{T}_n))\ E_{Q_{\overline{\mu}}}(g(\widehat{T}_l)).$$

ii) Applying the above result with $T'_{\mathbf{i}}$ instead of $T_{\mathbf{i}}$, it can be shown that the $\widehat{T}'_n, (n\geq 1)$ are $Q_{\overline{\nu}}$ -i.i.d. and that $E_{Q_{\overline{\nu}}} f(\widehat{T}'_n) = E\sum_{k=1}^{N}\frac{T_k}{S}f(\frac{T_k}{S})$.

iii) It can be shown that $\widehat{T}_n, (n\geq 1)$ are $Q_{\overline{\nu}}$-i.i.d. and that $E_{Q_{\overline{\nu}}} f(\widehat{T}_n) = E(\frac{1}{S}\sum_{k=1}^{N} T_k f(T_k))$, using the same argument as in i):

$$E_{Q_{\overline{\nu}}} f(\widehat{T}_n) = E\sum_{\mathbf{i}\in z_n} f(T_{\mathbf{i}})X'_{\mathbf{i}} = E\big(\sum_{\mathbf{i}\in z_{n-1}} X'_{\mathbf{i}}\ [\sum_{k=1}^{N_{\mathbf{i}}} f(T_{\mathbf{i}k})T'_{\mathbf{i}k}]\big)$$

$$= E[\sum_{\mathbf{i}\in z_{n-1}} X'_{\mathbf{i}}]\ E(\frac{1}{S}\sum_{k=1}^{N} T_k f(T_k)) = E(\frac{1}{S}\sum_{k=1}^{N} T_k f(T_k)).$$

In the same way, for $n < l$,

$$E_{Q_{\overline{\nu}}}[f(\widehat{T}_n)g(\widehat{T}_l)] = E\sum_{\mathbf{i}\in z_l} f(T_{\mathbf{i}|n})g(T_{\mathbf{i}})X'_{\mathbf{i}}$$

$$= E\big[\sum_{\mathbf{i}\in z_{l-1}} f(T_{\mathbf{i}|n})X'_{\mathbf{i}}\ \big(\sum_{k=1}^{N_{\mathbf{i}}} g(T_{\mathbf{i}k})T'_{\mathbf{i}k}\big)\big] = E[\sum_{\mathbf{i}\in z_{l-1}} f(T_{\mathbf{i}|n})X'_{\mathbf{i}}]\ E(\frac{1}{S}\sum_{k=1}^{N} T_k g(T_k))$$

$$= E[\sum_{\mathbf{i}\in z_n} f(T_{\mathbf{i}})X'_{\mathbf{i}}]\ E(\frac{1}{S}\sum_{k=1}^{N} T_k g(T_k)) = E_{Q_{\overline{\nu}}}(f(\widehat{T}_n))\ E_{Q_{\overline{\nu}}}(g(\widehat{T}_l)).$$

iv) With the same argument as in i) it can be shown that the $\widehat{T}'_n$ $(n\geq 1)$ are $Q_{\overline{\mu}}$-i.i.d. and that $E_{Q_{\overline{\mu}}} f(\widehat{T}'_n) = E(\sum_{i=1}^{N} T_i f(\frac{T_i}{S}))$.

v) Let us prove (11). We have :

$$E_{Q_{\overline{\mu}}} f(\widehat{W}_n) = E \sum_{\mathbf{i} \in z_n} f(\overline{W}_{\mathbf{i}}) X_{\mathbf{i}} \overline{W}_{\mathbf{i}} = E[\sum_{\mathbf{i} \in z_n} X_{\mathbf{i}}] \; E[\overline{W} f(\overline{W})] = E[\overline{W} f(\overline{W})],$$

and

$$E_{Q_{\overline{\nu}}} f(\widehat{W}_n) = E \sum_{\mathbf{i} \in z_n} f(\overline{W}_{\mathbf{i}}) X'_{\mathbf{i}} = E[\sum_{\mathbf{i} \in z_n} X'_{\mathbf{i}}] E[f(\overline{W})] = E[f(\overline{W})].$$

This ends the proof of lemma 10.

The two following lemmas give the precise asymptotical behaviour of X, X' and $\overline{W}$.

Lemma 11 - *Under assumption (H), as $n \to \infty$,*

$$\lim \frac{\log X_{\mathbf{i}|n}}{n} = E(\sum_{i=1}^{N} T_i \log T_i) \qquad Q_{\overline{\mu}} \quad a.s.,$$

$$\lim \frac{\log X_{\mathbf{i}|n}}{n} = E(\frac{1}{S} \sum_{i=1}^{N} T_i \log T_i) \qquad Q_{\overline{\nu}} \quad a.s.,$$

$$\lim \frac{\log X'_{\mathbf{i}|n}}{n} = E(\sum_{i=1}^{N} T_i \log T_i) - E(S \log S) \qquad Q_{\overline{\mu}} \quad a.s.,$$

$$\lim \frac{\log X'_{\mathbf{i}|n}}{n} = E(\frac{1}{S} \sum_{i=1}^{N} T_i \log T_i) - E(\log S) \qquad Q_{\overline{\nu}} \quad a.s..$$

Lemma 11 is proved applying the law of large numbers and lemma 10.

Lemma 12 - *Under assumption (H), as $n \to \infty$,*

(12) $$\lim \frac{\log \overline{W}_{\mathbf{i}|n}}{n} = 0 \qquad Q_{\overline{\mu}} \quad a.s., \qquad if \;\; (H_1),$$

(13) $$\lim \frac{\log \overline{W}_{\mathbf{i}|n}}{n} = 0 \qquad Q_{\overline{\nu}} \quad a.s., \qquad if \;\; (H_2).$$

Proof of Lemma 12 : From lemma 10, for every $\epsilon > 0$,

$$Q_{\overline{\mu}}(| \log \overline{W}_{\mathbf{i}|n} |> n\epsilon) = Q_{\overline{\mu}}(\overline{W}_{\mathbf{i}|n} > e^{n\epsilon}) + Q_{\overline{\mu}}(\overline{W}_{\mathbf{i}|n} < e^{-n\epsilon}) =$$

$$= E(\overline{W} 1_{\overline{W} > e^{n\epsilon}}) + E(\overline{W} 1_{\overline{W} < e^{-n\epsilon}}) \leq E(\overline{W} 1_{\overline{W} > e^{n\epsilon}}) + e^{-n\epsilon}.$$

If $E(\overline{W} \log^+ \overline{W}) < \infty$ we conclude that $\sum_{n=1}^{\infty} Q_{\overline{\mu}}(| \log W_{\mathbf{i}|n} |> n\epsilon) < \infty$, and hence by Borel-Cantelli, that $\limsup_n \frac{| \log W_{\mathbf{i}|n} |}{n} \leq \epsilon, \quad Q_{\overline{\mu}}$ a.s.. This proves (12).

In the same way, (13) can be proved noticing that

$$Q_{\overline{\nu}}(|\log \overline{W}_{\mathbf{i}|n}| > n\epsilon) = P(\overline{W} > e^{n\epsilon}) + P(\overline{W} < e^{-n\epsilon}),$$

$$\sum_{n=1}^{\infty} P(\overline{W} > e^{n\epsilon}) \leq \sum_{n=1}^{\infty} e^{-n\epsilon} < \infty$$

and

$$\sum_{n=1}^{\infty} E(1_{\overline{W}<e^{-n\epsilon}}) = O(E(1_{\overline{W}<1} \log \frac{1}{\overline{W}})) = O(E(|\log \overline{W}|)).$$

The following lemma is crucial for the study of $\overline{\mu}_\omega$.

Lemma 13 - *Under assumption (H), as $n \to \infty$,*

$$\overline{\mu}_\omega(B(\mathbf{i}|n)) \downarrow 0 \qquad Q_{\overline{\mu}} \text{ a.s..}$$

Proof of Lemma 13: Since the sequence is non increasing, it is sufficient to prove that $\overline{\mu}_\omega(B(\mathbf{i}|n) \to 0$ in $Q_{\overline{\mu}}$-measure . For every $\epsilon > 0$ and $\lambda < 1$ we have :

$$Q_{\overline{\mu}}(\overline{\mu}^0_\omega(B(\mathbf{i}|n) > \epsilon) = Q_{\overline{\mu}}(X_{\mathbf{i}|n}\overline{W}_{\mathbf{i}|n} > \epsilon)$$

$$= E\Big(\sum_{\mathbf{j}\in \mathbf{z_n}} X_{\mathbf{j}}\overline{W}_{\mathbf{j}} 1_{\,X_{\mathbf{j}}\overline{W}_{\mathbf{j}} > \epsilon}\Big)$$

$$= E\Big[\sum_{\mathbf{j}\in \mathbf{z_n}} X_{\mathbf{j}}\overline{W}_{\mathbf{j}} 1_{\,X_{\mathbf{j}}\overline{W}_{\mathbf{j}} > \epsilon} (1_{\,X_{\mathbf{j}} < \lambda^n} + 1_{\,X_{\mathbf{j}} \geq \lambda^n})\Big]$$

$$\leq E\Big(\sum_{\mathbf{j}\in \mathbf{z_n}} X_{\mathbf{j}}\overline{W}_{\mathbf{j}}\, 1_{\,\lambda^n\overline{W}_{\mathbf{j}} > \epsilon}\Big) + E\Big(\sum_{\mathbf{j}\in \mathbf{z_n}} X_{\mathbf{j}}\overline{W}_{\mathbf{j}} 1_{\,X_{\mathbf{j}} \geq \lambda^n}\Big)$$

$$= E(\overline{W} 1_{\,\overline{W} > \epsilon\lambda^{-n}}) + E_{Q_{\overline{\mu}}}(1_{\,X_{j|n} \geq \lambda^n})$$

$$= E(\overline{W}\; 1_{\,\overline{W} > \epsilon\lambda^{-n}}) + Q_{\overline{\mu}}(\frac{\log X_{\mathbf{j}|n}}{n} \geq \log \lambda).$$

Choosing $1 > \lambda > \exp(E(\sum_{k=1}^N T_k \log T_k))$, we see that the first term tends to 0 (easy) and from lemma 10 that the second term tends also to 0. Hence $\overline{\mu}^0_\omega(B(\mathbf{i}|n)) \to 0$ in $Q_{\overline{\mu}}$-measure, hence $\overline{\mu}_\omega(B(\mathbf{i}|n)) \downarrow 0$ a.s..

Proof of Theorem 6

1) Non-atomicity is a direct consequence of lemma 13: for a.e. ω, $\overline{\mu}_\omega(B(\mathbf{i}|n) \downarrow 0$ for $\overline{\mu}^0_\omega$-a.e. $\mathbf{i}$, hence for $\overline{\mu}_\omega$-a.e. $\mathbf{i}$. Now $\overline{\mu}_\omega(B(\mathbf{i}|n) \downarrow \overline{\mu}_\omega\{\mathbf{i}\}$ for every $\mathbf{i} \in \partial\omega$ and we may conclude. The same for $\overline{\nu}$ with T'_k instead of T_k.
2) If $S \equiv 1$ a.s., the two measures are identical. In the opposite case, following the sketch of proof of theorem 2, it can be shown that the two measures are mutually singular. Using the Hellinger distance between $\overline{\mu}_\omega$ and $\overline{\nu}_\omega$ we have: $\overline{\rho}_n(\omega) = \sum_{\mathbf{i}\in z_n} \sqrt{\overline{\mu}_\omega(B(\mathbf{i}))\overline{\nu}_\omega(B(\mathbf{i}))}$, and:

$$E(\overline{\rho}_n\sqrt{\overline{W}}) = E\left(\sum_{\mathbf{i}\in z_n} \sqrt{X_{\mathbf{i}}\overline{W}_{\mathbf{i}}X'_{\mathbf{i}}}\right) = E\sqrt{\overline{W}}\;(E\sqrt{S})^n.$$

As $E\sqrt{S} < \sqrt{ES} = 1$, we have $E(\overline{\rho}_n \sqrt{\overline{W}}) \to 0$ and then $\overline{\rho}_n \to 0$ a.s. since $\overline{\rho}_n$ is non-increasing. This ends the proof.

Proof of Theorem 7

We have $\log \overline{\mu}_\omega(B(\mathbf{i}|n) = \log X_{\mathbf{i}|n} + \log \overline{W}_{\mathbf{i}|n}$. From lemmas 11 et 12, we get:

$$\frac{\log \overline{\mu}_\omega(B(\mathbf{i}|n)}{n} \to E(\sum_{k=1}^{N} T_k \log T_k) \qquad Q_{\overline{\mu}} \quad a.s., \qquad if\ (H_1),$$

$$\frac{\log \overline{\mu}_\omega(B(\mathbf{i}|n)}{n} \to E(\frac{1}{S}\sum_{k=1}^{N} T_k \log T_k) \qquad Q_{\overline{\nu}} \quad a.s., \quad if\ (H_2).$$

This proves (5) and (7). Formulas (6) and (8) are obtained in the same way.

Proof of Theorem 3 (2),(3),(4) and Corollary 4

From theorem 7 and corollary 8 (applied to $T_k = 1/N$), it is sufficient to prove that $E \mid \log W \mid < \infty$. It is known that if $p_1 = 0$, then for every $k > 0$, and $x > 0$ small enough,

$$P(W < x) \leq x^k$$

(see for instance Biggins et Bingham (1993) th.3) and that if $p_1 > 0$ there exists a constant C such that for every $x > 0$ small enough,

$$P(W < x) \leq C x^\alpha$$

where $\alpha = -\log p_1 / \log m > 0$. (Biggins and Bingham (1993) th.4 or Dubuc (1971 th.1)). In both cases, there is some $\lambda > 0$ such that for every $x > 0$ small enough:

$$P(W < x) \leq x^\lambda.$$

This implies $\sum P(W < a^n) < \infty$ for every $a \in]0,1[$, and then $E[(\log 1/W) 1_{W<1}] < \infty$ or $E \mid \log W \mid < \infty$, and the proof is done.

References

Athreya, K.B. and Ney, P.E. (1972) Branching Processes, Springer Verlag, Berlin.

Biggins, J.D. (1977) Martingale convergence in the branching random walk, *J. Appl. Probab.* **14**, $25-37$.

Biggins, J.D. and Bingham, N.H. (1993) Large deviations in the supercritical branching process, *Adv. Appl. Prob.*$\mathbf{25}, 757-772$.

Billingsley, P. (1965) Ergodic Theory and Information, Wiley, New York.

Dacunha-Castelle, D. et Duflo, M. (1983) Probabilités et Statistiques, 2. Problèmes à temps mobile, Masson, Paris.

Dubuc, S. (1971) La densité de la loi limite d'un processus en cascade expansif, *Z.Warscheinlichkeitstheorie* **19**, 281-290.

Falconer, K.J. (1986) Random Fractals, *Math. Proc. Camb. Phil. Soc.* $\mathbf{100}, 559-582$.

Graf, S., Mauldin, R.D. and Williams, S.C. (1988) The exact Hausdorff dimension in random recursive constructions, *Mem. Amer. Math. Soc.*, **71** n^o381.

Hawkes, J. (1981) Trees generated by a simple branching process. *J. London Math. Soc.* $(2)\mathbf{24}, 373-384$.

Joffe, A. (1978) Remarks on the structure of trees with applications to supercritical Galton-Watson processes. *In Advances in Prob.* **5**, ed. A.Joffe and P.Ney, Dekker, New- York p.$263-268$.

Kahane, J.P. et Peyrière, J. (1976) Sur certaines martingales de Mandelbrot. *Adv. in Math.,* $\mathbf{22}, 131-145$.

Liu, Q. (1992) The exact Hausdorff dimensions of a branching set. *Prépublication n^o135, Laboratoire de Probabilités, Université Paris VI.*

Liu, Q. (1993) Sur quelques problèmes à propos des processus de branchement, des flots dans les réseaux et des mesures de Hausdorff associées. *Thèse, Université Paris VI, Laboratoire de Probabilités.*

Liu, Q. (1994) Sur une équation fonctionnelle et ses applications (I): une extension du théorème de Kesten-Stigum concernant des processus de branchement. *Prépublication de l'IRMAR*

Lyons, R., A simple path to Biggins' martingale convergence for branching random walk. *This volume*

Lyons, R., Pemantle R. and Peres Y. (1994) Ergodic theory on Galton-Watson trees, I: Speed of random walk and dimension of harmonic measure, *to appear in Ergodic Theory Dyn. Systems.*

Mauldin, R.D. and Williams, S.C. (1986) Random constructions, asymptotic geometric and topological properties, *Trans. Amer. Math. Soc.* $\mathbf{295}, 325-346$.

Neveu, J. (1986) Arbres et Processus de Galton-Watson. *Ann. IHP* **22**.

O'Brien, G.L. (1980) A limit theorem for sample maxima and heavy branches in Galton-Watson trees. *J. Appl. Prob.* $\mathbf{17}, 539-545$.